ACYCLIC COLOURING OF SOME OPERATIONS ON CERTAIN GRAPHS

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Abstract: *The aim of this paper is to discuss acyclic vertex colouring and to determine acyclic chromatic number of some operations like strong product, cartesian product, identifying vertices on certain graphs.*

Keywords: *Strong product of graphs, Cartesian product of graphs, Line graph, Ayclic colouring, Acyclic chromatic number.*

1. INTRODUCTION

In graph theory, *graph colouring* [2]-[4]-[7]-[10] is a special case of graph labeling; it is an assignment of labels traditionally called "colours" to elements of a graph subject to certain constraints. The most common types of colourings are vertex colouring, edge colouring and face colouring. A *k-vertex colouring* of a graph G is an assignment of k colours to the vertices of G and it is proper if no two adjacent vertices are assigned the same colour.

A proper vertex colouring of a graph is *acyclic* if every cycle uses at least three colour [8]. The *acyclic chromatic number* of G, denoted by $a(G)$, is the minimum k such that G admits an acyclic k-colouring.

In this paper we establish the acyclic chromatic number of $P_k \otimes P_n$ for $k = 2,3$, $L(P_2 \times P_n)$ and of some special graphs.

2. ACYCLIC CHROMATIC NUMBER OF STRONG PRODUCT OF TWO GRAPHS

2.1. Definition [5]**:** The *strong product* $G_1 \otimes G_2$ of two graphs G_1 and G_2 with disjoint point sets V_1 and V_2 and edge sets E_1 and E_2 is the graph with point set $V_1 \times V_2$. Then $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are adjacent in $G = G_1 \otimes G_2$ whenever $[u_1 = v_1$ and u_2 adj $v_2]$ or $[u_2 = v_2$ and u_1 adj $v_1]$ or $[u_1$ adj v_1 and u_2 adj $v_2]$.

2.2. Theorem: For $n \geq 2$, the acyclic chromatic number $a(P_2 \otimes P_n) = 4$.

Proof: Let $G = P_2 \otimes P_n$ be the strong product of two paths P_2 and P_n, where P_n is a path on n - vertices. Let $V(G) = \{v_1, v_2, v_3, \ldots, v_n, u_n, u_{n-1}, u_{n-2}, \ldots, u_1\}$ be the vertex

ISBN - 978-93-81583-57-9

set of G, which are named in anti- clock wise direction. Consider the colour class $C=\{c_1,c_2,c_3,c_4\}$. Assign the colour c_i to the vertices as follows.

For odd values of i, the colours c_1 and c_4 are assigned to v_i and u_i respectively. Similarly for even values of i, the colours c_2 and c_3 are assigned to v_i and u_i respectively. Now we prove that the colouring is acyclic. That is the colouring does not induce a bi-chromatic cycle. The colouring is in such a way that the subgraph induced by each $< c_1,c_2 >$, $< c_1,c_3 >$, $< c_2,c_4 >$ and $< c_3,c_4 >$ is a path P_n. Now we need to examine the subgraphs induced by $< c_1,c_4 >$ and $< c_2,c_3 >$.

Case 1. Subgraph induced by $< c_1,c_4 >$.

The subgraph induced by $< c_1,c_4 >$ is the disjoint union of paths P_2 and the numbers of such paths are given by

$\frac{n}{2}$, if n is even and

$\left\lceil \frac{n}{2} \right\rceil$, if n is odd,

which forms a forest.

Case 2. Subgraph induced by $< c_2,c_3 >$

The subgraph induced by $< c_2,c_3 >$ is again a disjoint union of paths P_2 and the numbers of such paths are given by

$\frac{n}{2}$, if n is even and

$\left\lfloor \frac{n}{2} \right\rfloor$, if n is odd,

which is also a forest.

Or in each case we can easily verify the result $\varepsilon = v - \omega$, [2] (ω is the number of components and ε is the number of edges) which is the necessary and sufficient condition for a forest.

Thus any pair of the colour class will never induce a bi- chromatic cycle in the graph. So the above said colouring is acyclic.

Also the colouring is minimum, as $G = P_2 \otimes P_n$ contains the subgraph K_4, minimum 4 colours are required for its proper colouring.

Hence $a(P_2 \otimes P_n) = 4$ for $n \geq 2$.

Example

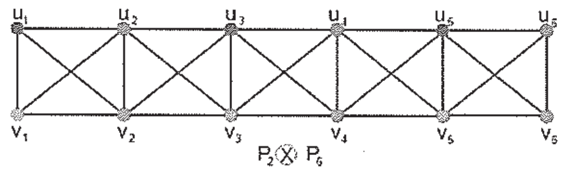

Fig. 1

Here, $a(P_2 \otimes P_6) = 4$.

Remark: $a(P_2 \otimes P_n) = 2$ for $n = 1$.

2.3 Corollary: For $n \geq 2$, the acyclic chromatic number of $P_3 \otimes P_n$ is 4.

3. ACYCLIC CHROMATIC NUMBER OF A SPECIAL GRAPH.

3.1 Theorem: Let G be the connected graph, which is obtained by sequentially identifying n-copies $(n \in N)$ of the octahedral graphs T^j such that one vertex of T^j is identified with T^{j+1} for $j = 1,2,3,...,n-1$, then the acyclic chromatic number, $a(G) = 5$.

Some Structural properties of G, for $n > 1$.

- Number of vertices $V = 5n+1$.
- Number of edges $\varepsilon = 12n$.
- Maximum Degree of the vertices Δ = 8, for $n > 1$
- Minimum Degree of vertices δ = 4.

Proof: It is given that G contains n copies of octahedral graphs T^j for $j = 1,2,3,...,n$ such that each pair of the consecutive subgraphs T^j has only a single vertex in common. Let $V(G) = \{u_i^j\}$ for $1 \leq i \leq 6$, $j = 1,2,3,...,n$ be the vertex set of G such that T^j and T^{j+1} has the common vertex $v_1^j = v_1^{j+1}$ for odd j, and has the common vertex $v_3^j = v_3^{j+1}$ for even j. Consider the colour class

$C=\{c_i\}$ for $1 \le i \le 5$. Assign the colour c_5 to the common vertices v_1^j and assign the colour c_3 to the common vertices v_3^j for all j. The colours c_2 and c_4 are assigned to the vertices v_2^j and v_4^j respectively. The rest are assigned the colour c_1.

Now by [6, 11], suggested by Robert E. Jamison, the acyclic chromatic number of a graph G satisfies $a(G) > \frac{|E(G)|}{|V(G)|}+1$.

So in our case it is $a(G) > \frac{12n}{5n+1}+1$

Hence $a(G) > 2.4+1$

This implies

$a(G) \ge 4$,

Also since G is a planar graph, by Borodin [3], $a(G) \le 5$. That is $4 \le a(G) \le 5$. If we replace any one of the colour by an already used colour, then either the colouring become improper or cyclic. So that our colouring is minimum. Now we prove that the colouring is acyclic. That is the colouring does not induce a bi-chromatic cycle. Since each pair of the octahedral graph T^j for $j=1,2,3,\ldots,n$ has only a single vertex in common, no cycle is possible through different T^j, so the cycle should be in a single T^j itself. Now this cycle is not bi-chromatic as the only colour which occur twice is c_1 and all other c_i for $i=2,3,4,5$ occur only once in any T^j. Hence the colouring is acyclic.

Thus $a(G)=5$ for $n \in N$.

Example

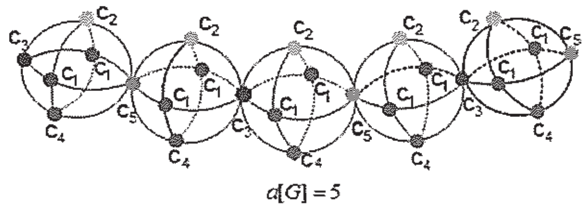

Fig. 2

4. ACYCLIC CHROMATIC NUMBER OF LINE GRAPH

4.1 Definition [5]: Let G be a finite undirected graph with no loops and multiple edges, the *line graph* of G, denoted by $L(G)$, is the intersection graph $\Omega(X)$. Thus the points of $L(G)$ are the lines of G, with two points of $L(G)$ are adjacent whenever the corresponding lines of G are.

4.2 Definition[5]: The cartesian graph product $G = G_1 \times G_2$, sometimes simply called the graph product of graphs G_1 and G_2 with disjoint point sets V_1 and V_2 and edge sets E_1 and E_2 is the graph with point set $V_1 \times V_2$. Then $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are adjacent in $G = G_1 \times G_2$ whenever $[u_1 = v_1$ and $u_2 \text{ adj } v_2]$ or $[u_2 = v_2$ and $u_1 \text{ adj } v_1]$.

4.3 Observation: The acyclic chromatic number, $a(L[P_2 \times P_n]) = 3$ for $n \geq 2$

Example

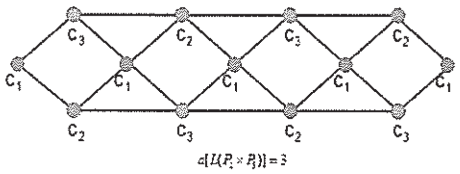

Fig. 3

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