ON THE IFS GENERATING SUPER SELF SIMILAR SETS

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***Abstract:** A hyperbolic Iterated Function System (IFS) consists of a complete metric space (X, d) together with a finite set of contraction mappings on X and self similarity is a property which defines a fractal. In this article, we focus on IFS generating super self similar sets. A non empty compact set is super self similar if it contains a union of scaled copies of itself, with scaling factor less than one. We have discussed the type of IFS leading to sub self similar and super self similar sets. We have also analysed their properties when a scaling is applied to the IFS.*

***Keywords:** self similar sets, sub self similar sets, super self similar sets, iterated function system.*

1. Introduction

Fractals are non-regular geometric shapes that have the same degree of non-regularity on all scales. A general account of fractals was given by Mandelbrot [1], whilst a more mathematical approach may be found in Falconer [6]. A wide range of fractals are self similar in the sense that they are made up of arbitrarily small copies of themselves. For example, the middle third Cantor set is the union of two similar copies, and the Von Koch Curve is made up of four similar copies.

In order to understand this type of fractals it is important to know about contracting similarities. The self similarities are not only properties of the fractals; they may actually be used to define them. Self similar sets were presented in a unified way by Hutchinson [3] and the calculation of their dimensions and measures; see for example [3, 4, 5]. Sub self similar sets and super self similar sets were defined by Falconer [4, 5].

As Barnsley, Demko and others have shown [9, 10, 11, 12], an effective method for producing fractal shapes (in any number of dimensions) is with Iterated Function Systems (IFSs), using the "Chaos Game" algorithm (or some deterministic algorithm).

This approach has been used for producing naturalistic shapes [11], finding fractal interpolants to given data [12] and fractal approximations of given functions [13], and even for visualizing arbitrary discrete sequences [14]. Indeed, any contractive IFS will give an attractor (usually of fractal dimension); thus it is possible to generate IFSs at random to explore the graphical possibilities. Similarly, because the attractor depends continuously on the parameters in the IFS [10], small data sets from any source could be encoded as IFSs for visualization.

This paper is organised in the following manner. We review some preliminaries in the second section. The main results are discussed in the third and fourth sections.

ISBN - 978-93-81583-57-9

2. PRELIMINARIES

We work in a fixed Euclidean space R^n. Let H (R^n) denote the set of non-empty, compact subsets of R^n.

Definition2.1: [10] The Hausdorff metric h on H (R^n) is defined by

$$h(A, B) = \max\{\sup_{x\in A}\{dist(x, B)\}, \sup_{y\in B}\{dist(y, A)\}\}$$

where $A, B \in H(R^n)$

We can see that $H(R^n)$ is complete in the metric h.

Definition 2.2: [6] A mapping $S : R^n \to R^n$ is called a contraction on R^n if there is a number c with $0 < c < 1$ such that $\|S(x) - S(y)\| \le c\|x - y\|, \forall x, y \in R^n$. Clearly, any contraction is a continuous mapping. If equality holds, ie; if $\|S(x) - S(y)\| = c\|x - y\|$, then S transforms sets into geometrically similar ones and we call S a similarity.

Definition 2.3: [10] A (hyperbolic) IFS consists of a complete metric space (X, d) together with a finite set of contraction mappings $w_j : X \to X$ with respect to the contraction factors s_j, for $j = 1.2,...N$ The notation of the IFS just defined is $\{X; w_j, j = 1,2,...N\}$ and its contraction factor is $s = \max\{s_j : j = 1,2,...N\}$.

Theorem 2.4: [10] Let $\{X; w_j, j = 1,2,...N\}$ be a hyperbolic IFS with contraction factor *s*. Then the transformation $W : H(X) \to H(X)$ defined by $W(B) = \bigcup_{j=1}^{N} w_j(B)$ for all $B \in H(X)$, is a contraction mapping on the complete metric space $(H(X), h)$ with contraction factor *s*. That is, $h(W(B), W(C)) \le s.h(B, C)$ for all $B, C \in H(X)$. Its unique fixed point, $A \in H(X)$ obeys $A = W(A) = \bigcup_{j=1}^{N} w_j(A)$ and is given by $A = \lim_{n\to\infty} W^{on}(B)$ for any $B \in H(X)$.

Definition 2.5: [10] The fixed point $A \in H(X)$ described in the Theorem 2.4 is called the attractor of the IFS.

There are many classes of IFS of special interest. If the $\{w_j, j = 1,2,...N\}$ are similarities, the attractor E is called self-similar, if they are affine transformations E is called self-affine, and if they are conformal transformations then E is called self-conformal.

Definition 2.6: [4] Given m > 2, and contracting similarities $S_i : R^n \rightarrow R^n$, (i =1, 2, 3, . . . , m) there exists a unique non-empty compact set $E \subseteq R^n$ satisfying

$$E = \bigcup_{i=1}^{m} S_i(E) \tag{1}$$

This set E is called self-similar, or self-similar for $\{S_1, . . ., S_m\}$ if the similarity transformations need to be emphasized. Thus a self-similar set is a metric space that is the union of scaled versions of itself, with scaling factor less than one [8].

In his paper on sub self-similar sets, Falconer gives a generalisation of self-similar sets by relaxing the inequality in eq(1) to inclusion. ie; With m > 2, and contracting similarities $S_i : R^n \longrightarrow R^n$, (i = 1, 2, 3, . . . , m), a non-empty compact set $E \subseteq R^n$ is called sub self-similar for $\{S_1, . . . , S_m\}$, if

$$E \subseteq \bigcup_{i=1}^{m} S_i(E) \tag{2}$$

Later, Falconer also defined super self similar set by reversing the inclusion. A non-empty compact set $E \subseteq R^n$ is called super self similar, if there are contracting similarities $S_i : R^n \rightarrow R^n$, (i = 1, 2, 3, . . . , m) for m > 2 such that

$$E \supseteq \bigcup_{i=1}^{m} S_i(E) \tag{3}$$

According to Falconer, no finite sets in $H(R^n)$ with more than one element is super self similar. And since finite sets are dense in $H(R^n)$, the set of non super self similar sets is dense in $H(R^n)$.

3. CONTRACTING SIMILARITIES

It is clear from the definition of contracting similarity transformation that it is a composition of isometry and homothety. A transformation which preserves distances is called an isometry. A transformation $\mu_c : R^n \rightarrow R^n$ is a homothety if $\mu_c(x) = cx (c \geq 0)$. A transformation $\tau_b : R^n \rightarrow R^n$ is a translation if $\tau_b(x) = x - b$. Translation is an isometry.

Proposition 3.1. [3] A transformation $S : R^n \rightarrow R^n$ is a similarity transformation iff $S = \mu_c o \tau_b o O$ for some homothety μ_c, translation τ_b, and orthonormal transformation O.

In a contraction, the scaling factor c is such that $0 < c < 1$. Thus a contracting similarity can transform points of a given set to points inside or outside that set. Accordingly we make the following definitions.

Definition 3.2. If a contracting similarity maps points of a given set E to points which lie on the boundary or interior of E, then we call it a contracting sub similarity transformation.

Definition 3.3. If a contracting similarity maps points of a given set E to points which also lies outside E, then we call it a contracting super similarity transformation.

Thus contracting sub similarities maps a set into itself whereas contract-ing super similarities maps a set to its outside. For example, consider the set E = [0, 1] then the transformation $S_1 = \frac{1}{2}x$ maps E to $[0, \frac{1}{2}]$ and hence is a contracting sub similarity, whereas the transformation $S_2 = \frac{1}{2}x + 1$ maps E to $[1, \frac{3}{2}]$ which contains points outside E and hence is a contracting super similarity.

Now we define strictly sub self similar and strictly super self similar sets.

Definition 3.4. A non-empty compact set $E \subseteq R^n$ is called strictly sub self similar, if there are contracting similarities $S_i : R^n \to R^n$, (i = 1, 2, 3, . . . , m) for m > 2 such that

$$E \subset \bigcup_{i=1}^{m} S_i(E) \tag{4}$$

and is strictly super self similar if we reverse the set inclusion in eq. (4).

This leads to an interesting classification of contracting similarities which define sub self similar sets and super self similar sets. For a set to be sub self similar, the contracting similarities which define the set should contain only contracting super similarities. In the similar way, for super self similar sets, the associated similarities should be contracting sub similarities. We summarize these results in the following theorem.

Theorem 3.5. A non empty compact subset $E \subseteq R^n$ is strictly sub self similar for the contracting similarities $S_i : R^n \to R^n$, (i = 1, 2, 3, . . . , m), for m > 2, iff S_i, for$1 \leq i \leq m$ are contracting super similarities. Similarily, a non empty compact subset $E \subseteq R^n$ is strictly super self similar for the contracting similarities $S_i : R^n \to R^n$, (i = 1, 2, 3, . . . , m), for m > 2, iff S_i, for$1 \leq i \leq m$ are contracting sub similarities.

Proof: By definition, a non empty compact subset $E \subseteq R^n$ is strictly sub self similar for the contracting similarities $S_i : R^n \to R^n$, (i = 1, 2, 3, . . . , m), for m > 2, if $E \subset \bigcup_{i=1}^{m} S_i(E)$.

This is possible only if each S_i, for $1 \le i \le m$ maps points of E to points which may also lie outside of E, ie; each S_i is a contracting super similarity.

Conversely, let each S_i, for $1 \le i \le m$ be a contracting super similarity. Then $S_i(E)$ contains E for $1 \le i \le m$ and hence $\bigcup_{i=1}^{m} S_i(E)$ contains E which says E is strictly sub self similar.

The proof for strictly super self similar set can be obtained in a similar way.

4. SCALED IFS

In this paper, we have also studied the nature of an IFS when a scaling is applied to it.

Theorem 4.1: Let $\{X; S_j, j = 1,2,..m\}$ be an IFS with contraction factor $s = \max\{s_j : j = 1,2,..m\}$ where $\{S_j = s_j x + t_j\}_{j=1}^{m}$ are similarity maps on X. Then $\{X; k_j S_j, j = 1,2,..m\}$ is an IFS if and only if $0 < k_j < \frac{1}{s_j}$ for $j = 1.2,..m$.

Proof: The similarity maps $\{S_j = s_j x + t_j\}_{j=1}^{m}$ on a complete metric space (X,d) is an IFS iff each of the map $S_j = s_j x + t_j$ is a contraction mapping on X. ie; $0 < s_j < 1$, for $j = 1.2,..m$.

Thus the similarity map $k_j S_j = k_j s_j x + k_j t_j$ is a contraction mapping iff $0 < k_j s_j < 1$. That is, iff $0 < k_j < \frac{1}{s_j}$ for $j = 1.2,..m$.

Definition 4.2: The IFS $\{X; k_j S_j, j = 1,2,...m\}$ defined as per the theorem above is called the scaled IFS.

We know that a set $E \subseteq R^n$ is sub self similar for the contracting similarities $S_i : R^n \rightarrow R^n$, (i = 1, 2, 3, . . . , m), for m > 2, if it satisfies eq.(2) in definition 2.6. Then $\{R^n; S_i, 1 \le i \le m\}$is the IFS which generates the set E. in In the following theorems, we give conditions for which scaling preserves sub self similarity and super self similarity.

Theorem 4.3: Let $E \subseteq X$ be the attractor of the IFS $\{X; S_i, 1 \le i \le m\}$ which is strictly sub self similar for the contracting similarities $S_i : X \rightarrow X$, (i = 1, 2, 3, . . . , m). Then the attractor E_1 of the scaled IFS $\{X; k_i S_i, 1 \le i \le m\}$is also a strictly sub self similar set, for those values of $k_i \in (0, \frac{1}{s_i})$ for which $k_i S_i$ is a contracting super similarity for each $j = 1.2,..m$.

Proof: Given Let $E \subseteq X$ is a sub self similar set. By theorem 3.5, each $S_i : X \rightarrow X$, (i = 1, 2, . . . , m) is a contracting super self similarity transformation.

We apply a scaling to S_i so that $k_i S_i$ is a contracting super similarity for i = 1, 2, . . . , m. Thus by theorem 3.5 the set E_1 is strictly sub self similar.

Theorem 4.4: Let $E \subseteq X$ be the attractor of the IFS $\{X; S_i, 1 \le i \le m\}$ which is super self similar for the contracting similarities $S_i : X \rightarrow X$, (i = 1, 2, 3, . . . , m). Then the attractor E_1 of the scaled IFS $\{X; k_i S_i, 1 \le i \le m\}$is also a super self similar set, for for those values of $k_i \in (0, \frac{1}{s_i})$ for which $k_i S_i$ is a contracting super similarity for each i = 1, 2, . . . , m.

Proof: The proof is similar to that of theorem 4.3

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