CONGRUENCE RELATION ON A*-ALGEBRAS

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Abstract: This paper presents theorems on congruence relation on A- algebras.*

Keywords: A-algebra, Sub-A*-algebra, Homomorphism, Isomorphism and Congruence.*

Definition 1: An algebra $(A, \wedge, *, (-)^{\sim}, (-)_{\pi}, 1)$ is an A*-algebra if it satisfies:

$a_\pi \vee (a_\pi)^{\sim}=1$, $(a_\pi)_\pi = a_\pi$ where $a \vee b = (a^{\sim} \wedge b^{\sim})^{\sim}$
$a_\pi \vee b_\pi = b_\pi \vee a_\pi$
$(a_\pi \vee b_\pi) \vee c_\pi = a_\pi \vee (b_\pi \vee c_\pi)$
$(a_\pi \wedge b_\pi) \vee (a_\pi \wedge (b_\pi)^{\sim}) = a_\pi$
$(a \wedge b)_\pi = a_\pi \wedge b_\pi$, $(a \wedge b)^{\#} = a^{\#} \vee b^{\#}$ where
$a^{\#} = (a_\pi \vee a^{\sim}{}_\pi)^{\sim}$
$a^{\sim}{}_\pi = (a_\pi \vee a^{\#})^{\sim}$, $a^{\sim\#} = a^{\#}$
$(a * b)_\pi = a_\pi$, $(a * b)^{\#} = (a_\pi)^{\sim} \wedge (b^{\sim}{}_\pi)^{\sim}$
$a = b$ if and only if $a_\pi = b_\pi$, $a^{\#} = b^{\#}$.

We write 0 for $1^{\sim}$, 2 for $0 * 1$.

Remark: The motivation for the operation $*$ is the "disjointifica-tion" of the familiar rectangular bands of semigroup theory which provide an equational way of the composing a set into a Cartesian product with two factors.

Example: 3={0, 1, 2} with the operations defined below is an A*- algebra.

$\wedge$	0	1	2
0	0	0	2
1	0	1	2
2	2	2	2

$\vee$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

$*$	0	1	2
0	0	2	2
1	1	1	1
2	0	2	2

x	0	1	2
$x^{\sim}$	1	0	2
x_π	0	1	0
$x^{\#}$	0	0	1

Note: From Definition 1 (i) through (iv) and Huntington's Theorem $B(A)= \{a_\pi \mid a \in A\}$ is a Boolean algebra with $\wedge$, $\vee$, $(-)^{\sim}$, 0 and $a \in B(A) \Rightarrow a_\pi=a$. Since 1, 0, $(a_\pi)^{\sim} \in B(A)$, we have $1_\pi=1$, $0_\pi=0$, $(a_\pi)^{\sim}{}_\pi=(a_\pi)^{\sim}$ and $a_\pi \wedge a^{\#}=0$, $a * 0=a_\pi$.

Lemma 1: For any x, y, z in an A*- algebra

$x^{\sim\sim} = x$

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$(x \wedge y)^{\sim}_{\pi}=(x^{\sim} \wedge y)_{\pi} \vee (x \wedge y^{\sim})_{\pi} \vee (x^{\sim} \wedge y^{\sim})_{\pi}$
$(x \vee y)_{\pi} =(x^{\sim} \wedge y)_{\pi} \vee (x \wedge y^{\sim})_{\pi} \vee (x \wedge y)_{\pi}$
$x \wedge (y \vee z)= (x \wedge y) \vee (x \wedge z)$.

Lemma 2: For any x,y in A
$(x * y)^{\sim}_{\pi}=(x_{\pi})^{\sim} \wedge (y^{\sim})_{\pi}$
$x = x_{\pi} * (x^{\sim})_{\pi}{}^{\sim}=(x_{\pi}) * x^{\#}$
If $x = e * f$, where $e, f \in B(A)$, $e \wedge f = 0$, then $x_{\pi} = e$, $x^{\#} = f$.

Theorem 1: Every A*-algebra $(A, \wedge, *, (-)_{\pi}, (-)^{\sim}, 1)$ satisfies the following conditions:
For x, y, z in A
$x \wedge (y \wedge z)= (x \wedge y) \wedge z$
$x \wedge y = y \wedge x$
$x \wedge x = x$
$1 \wedge x = x$
$x^{\sim\sim} = x$
$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ where $x \vee y =(x^{\sim} \wedge y^{\sim})^{\sim}$
$1_{\pi}= 1$
$[(x_{\pi})^{\sim}]_{\pi} = (x_{\pi})^{\sim}$
$(x \wedge y)_{\pi}= x_{\pi} \wedge y_{\pi}$
$(x \wedge x^{\sim})_{\pi}= 0$ where $1^{\sim}=0$
$x_{\pi} \wedge (x_{\pi} \vee y_{\pi})= x_{\pi}$
$(x \wedge y)^{\sim}_{\pi}=(x \wedge y^{\sim})_{\pi} \vee (x^{\sim} \wedge y)_{\pi} \vee (x^{\sim} \wedge y^{\sim})_{\pi}$
$(x_{\pi})_{\pi} = x_{\pi}$
$(x * y)_{\pi} = x_{\pi}$
$(x * y)^{\sim}_{\pi}=(x_{\pi})^{\sim} \wedge (y^{\sim})_{\pi}$
$x = x_{\pi} * (x^{\sim}{}_{\pi})^{\sim}$

E.G. Manes around 1989, in a rough draft of his paper entitled "The Equational Theory of Disjoint Alternatives", the algebra $(A, \wedge, *, (-)_{\pi}, (-)^{\sim}, 1)$ satisfying Th.1(i) through (xvi) called as an ada, which however differs from the definition of an ada.

Theorem 2: An algebra $(A, \wedge, *, (-)^{\sim}, (-)_{\pi}, 1)$ satisfying axioms of the above theorem is an A*-algebra.

Definition 2: Let $(A, \wedge, *, (-)^{\sim}, (-)_{\pi}, 1)$ be an A*-algebra and $A_1 \subseteq A$, A_1 is called a sub A*-algebra of A if A_1 is closed under $\wedge, *, (-)^{\sim}, (-)_{\pi}, 0, 1$.

Definition 3: Let $(A_1, \wedge, \vee, (-)^{\sim}, (-)_{\pi}, *, 1)$ and $(A_2, \wedge, \vee, (-)^{\sim}, (-)_{\pi}, *, 1)$ be A*- algebras. A mapping $f : A_1 \to A_2$ is called an A*-homomorphism if

(i) $f(a \wedge b)= f(a) \wedge f(b)$ (ii) $f(a * b)= f(a) * f(b)$
(iii) $f(a_{\pi})= (f(a))_{\pi}$ (iv) $f(a^{\sim})= (f(a))^{\sim}$

(v) $f(1) = 1$ (vi) $f(0) = 0$.

If in addition f is bijective, then f is called an A*-isomorphism, and A_1, A_2 are said to be isomorphic, denote in symbols $A_1 \cong A_2$.

Definition 4: A congruence relation $\emptyset$ on an A*-algebra is an equivalence relation on A satisfying

(i) $a \emptyset b \Rightarrow a_\pi \emptyset b_\pi$, $a^\# \emptyset b^\#$, $a^\sim \emptyset b^\sim$
(ii $a \emptyset b, c \emptyset d \Rightarrow (a * c) \emptyset (b * d), (a \wedge c) \emptyset (b \wedge d)$.

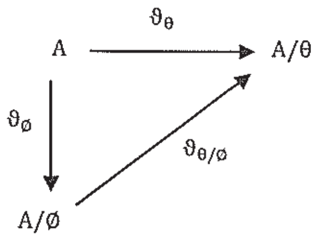
Note: Definition 4 is equivalent to

(i) $a \emptyset b \Rightarrow a^\sim \emptyset b^\sim$
(ii) $a \emptyset b, c \emptyset d \Rightarrow (a \wedge c) \emptyset (b \wedge d)$
(iii) $a \emptyset b, c \emptyset d \Rightarrow (a * c) \emptyset (b * d)$.

Theorem 3: Let $\emptyset$ be a congruence relation on an A*-algebra A. Then $A/\emptyset = \{\emptyset(a) \mid a \in A\}$ is an A*- algebra where operations are defined as follows:

(i) $\emptyset(a) \wedge \emptyset(b) = \emptyset(a \wedge b)$
(ii) $\emptyset(a)^\sim = \emptyset(a^\sim)$
(iii) $\emptyset(a)_\pi = \emptyset(a_\pi)$
(iv) $\emptyset(a) * \emptyset(b) = \emptyset(a * b)$.

Theorem 4: Suppose A is an A*-algebra and $\emptyset$ is a congruence relation on A. Suppose θ is another congruence on A. Define $\vartheta_{\theta/\emptyset} : A/\emptyset \to A/\theta$ as $\bar{a}_\emptyset \mapsto \bar{a}_\theta$. Then $\vartheta_{\theta/\emptyset}$ is a map if and only if $\emptyset \subseteq \theta$. So, in this case $\vartheta_{\theta/\emptyset}: \bar{a}_\emptyset \mapsto \bar{a}_\theta$ is a unique map such that



$\vartheta_{\theta/\emptyset}\vartheta_\emptyset = \vartheta_\theta$
is commutative.

Proof: Claim: $\vartheta_{\theta/\emptyset}$ is a map $\Leftrightarrow \emptyset \subseteq \theta$.

Suppose $\vartheta_{\theta/\emptyset} : A/\emptyset \to A/\theta$ by $\bar{a}_\emptyset \mapsto \bar{a}_\theta$ is a map.

$(a, b) \in \emptyset \Rightarrow \bar{a}_\emptyset = \bar{b}_\emptyset \Rightarrow \bar{a}_\theta = \bar{b}_\theta$ (since $\vartheta_{\theta/\emptyset}$ is a map)
$\Rightarrow (a, b) \in \theta$.

Therefore $\emptyset \subset \theta$.
Conversely suppose that $\emptyset \subset \theta$.

Claim: $\vartheta_{\theta/\emptyset} : A/\emptyset \to A/\theta$ by $\bar{a}_\emptyset \mapsto \bar{a}_\theta$ is a map.
Suppose $\bar{a}_\emptyset = \bar{b}_\emptyset \Rightarrow (a, b) \in \emptyset \Rightarrow (a, b) \in \theta$ $(\because \emptyset \subset \theta) \Rightarrow \bar{a}_\theta = \bar{b}_\theta$.
Therefore $\vartheta_{\theta/\emptyset}$ is well defined.Therefore $\vartheta_{\theta/\emptyset}$ is a map.
The diagram is commutative:

Let $a \in A$.
$(\vartheta_{\theta/\emptyset}\vartheta_\emptyset)(a) = \vartheta_{\theta/\emptyset}(\vartheta_\emptyset(a)) = \vartheta_{\theta/\emptyset}(\bar{a}_\emptyset) = \bar{a}_\theta = \vartheta_\theta(a)$
Therefore $\vartheta_{\theta/\emptyset}\vartheta_\emptyset = \vartheta_\theta$.
$\vartheta_{\theta/\emptyset}$ is unique:
Let $\vartheta : A/\emptyset \to A/\theta$ be a map such that $\vartheta\vartheta_\emptyset = \vartheta_\theta$.Let $\bar{a}_\emptyset \in A/\emptyset$.
$\vartheta_{\theta/\emptyset}(\bar{a}_\emptyset) = \vartheta_{\theta/\emptyset}(\vartheta_\emptyset(a)) = (\vartheta_{\theta/\emptyset}\vartheta_\emptyset)(a) = \vartheta_\theta(a) = (\vartheta\vartheta_\emptyset)(a)$
$= \vartheta(\vartheta_\emptyset(a)) = \vartheta(\bar{a}_\emptyset)$.

Therefore $\vartheta_{\theta/\emptyset} = \vartheta$.

Therefore $\vartheta_{\theta/\emptyset}$ is unique.
Claim: $\vartheta_{\theta/\emptyset}$ is a homomorphism.
Let $\bar{a}_\emptyset, \bar{b}_\emptyset \in A/\emptyset$.
$\vartheta_{\theta/\emptyset}(\bar{a}_\emptyset \wedge \bar{b}_\emptyset) = \vartheta_{\theta/\emptyset}((\overline{a \wedge b})_\emptyset) = (\overline{a \wedge b})_\theta = \bar{a}_\theta \wedge \bar{b}_\theta$
$= \vartheta_{\theta/\emptyset}(a) \wedge \vartheta_{\theta/\emptyset}(b)$
$\vartheta_{\theta/\emptyset}(\bar{a}_\emptyset * \bar{b}_\emptyset) = \vartheta_{\theta/\emptyset}((\overline{a * b})_\emptyset) = (\overline{a * b})_\theta = \bar{a}_\theta * \bar{b}_\theta$
$= \vartheta_{\theta/\emptyset}(a) * \vartheta_{\theta/\emptyset}(b)$
$\vartheta_{\theta/\emptyset}(\bar{a}_{\emptyset\pi}) = \vartheta_{\theta/\emptyset}(\bar{a}_{\pi\emptyset}) = \bar{a}_{\pi\theta} = \bar{a}_{\theta_\pi} = \vartheta_{\theta/\emptyset}(\bar{a}_\emptyset)_\pi$
$\vartheta_{\theta/\emptyset}(\bar{a}_\emptyset^{\sim}) = \vartheta_{\theta/\emptyset}(\overline{a^{\sim}}_\emptyset) = \overline{a^{\sim}}_\theta = \bar{a}_\theta^{\sim} = \vartheta_{\theta/\emptyset}(\bar{a}_\emptyset)^{\sim}$
and $\vartheta_{\theta/\emptyset}(\bar{0}_\emptyset) = \bar{0}_\theta$, $\vartheta_{\theta/\emptyset}(\bar{1}_\emptyset) = \bar{1}_\theta$, $\vartheta_{\theta/\emptyset}(\bar{2}_\emptyset) = \bar{2}_\theta$.

Therefore $\vartheta_{\theta/\square}$ is a homomorphism.
Clearly $\vartheta_{\theta/\square}$ is surjective.

Claim: $\vartheta_{\theta/\emptyset}$ is injective $\Leftrightarrow \theta = \emptyset$.
Suppose $\vartheta_{\theta/\emptyset}$ is an injective.

Claim: $\theta = \emptyset$.
Clearly $\emptyset \subset \theta$.
$(a, b) \in \theta \Rightarrow \bar{a}_\theta = \bar{b}_\theta \Rightarrow \vartheta_{\theta/\emptyset}(\bar{a}_\emptyset) = \vartheta_{\theta/\emptyset}(\bar{b}_\emptyset)$
$\Rightarrow \bar{a}_\emptyset = \bar{b}_\emptyset$ (since $\vartheta_{\theta/\emptyset}$ is an injective)
$\Rightarrow (a, b) \in \emptyset$.

Therefore $\theta \subset \emptyset$.

Therefore $\theta = \emptyset$.
Conversely suppose $\theta = \emptyset$.

<u>Claim</u>: $\vartheta_{\theta/\emptyset}$ is an injective.
Suppose $\vartheta_{\theta/\emptyset}(\bar{a}_\emptyset) = \vartheta_{\theta/\emptyset}(\bar{b}_\emptyset) \Rightarrow \bar{a}_\theta = \bar{b}_\theta \Rightarrow (a, b) \in \theta$

$$\Rightarrow (a, b) \in \emptyset \text{ (since } \theta = \emptyset)$$
$$\Rightarrow \bar{a}_\emptyset = \bar{b}_\emptyset.$$

Therefore $\vartheta_{\theta/\emptyset}$ is injective.

Note: $\theta/\emptyset = \{ \bar{a}_\emptyset \in A/\emptyset \mid \bar{a}_\theta = \bar{0}_\theta \}$. This is also called kernel of $\vartheta_{\theta/\emptyset}$.

Note: Define $\theta/\emptyset = \{(\bar{a}_\emptyset, \bar{b}_\emptyset) \mid \bar{a}_\emptyset, \bar{b}_\emptyset \in A/\emptyset \text{ and } \bar{a}_\theta = \bar{b}_\theta\}$. $\theta/\emptyset$ is also called kernel of $\vartheta_{\theta/\emptyset}$.

Theorem 5: $\theta/\emptyset$ is a congruence relation on $A/\emptyset$.

Proof: $(\bar{a}_\emptyset, \bar{b}_\emptyset) \in \theta/\emptyset \Leftrightarrow \bar{a}_\theta = \bar{b}_\theta$.
Clearly $\theta/\emptyset$ is an equivalence relation.

Suppose $(\bar{a}_\emptyset, \bar{b}_\emptyset) \in \theta/\emptyset$, $(\bar{c}_\emptyset, \bar{d}_\emptyset) \in \theta/\emptyset$.

$$\Rightarrow \bar{a}_\theta = \bar{b}_\theta, \bar{c}_\theta = \bar{d}_\theta$$
$$\Rightarrow a\ \theta\ b,\ c\ \theta\ d$$
$$\Rightarrow (a \wedge c)\ \theta (b \wedge d)$$
$$\Rightarrow (\overline{a \wedge c})_\theta = (\overline{b \wedge d})_\theta$$
$$\Rightarrow ((\overline{a \wedge c})_\emptyset, (\overline{b \wedge d})_\emptyset) \in \theta/\emptyset$$
$$\Rightarrow (\bar{a}_\emptyset \wedge \bar{c}_\emptyset, \bar{b}_\emptyset \wedge \bar{d}_\emptyset) \in \theta/\emptyset.$$

Suppose $(\bar{a}_\emptyset, \bar{b}_\emptyset) \in \theta/\emptyset$, $(\bar{c}_\emptyset, \bar{d}_\emptyset) \in \theta/\emptyset$.

$$\Rightarrow \bar{a}_\theta = \bar{b}_\theta, \bar{c}_\theta = \bar{d}_\theta \Rightarrow a\ \theta\ b,\ c\ \theta\ d$$
$$\Rightarrow (a * c)\ \theta (b * d) \quad \Rightarrow (\overline{a * c})_\theta = (\overline{b * d})_\theta$$
$$\Rightarrow ((\overline{a * c})_\emptyset, (\overline{b * d})_\emptyset) \in \theta/\emptyset$$
$$\Rightarrow (\bar{a}_\emptyset * \bar{c}_\emptyset, \bar{b}_\emptyset * \bar{d}_\emptyset) \in \theta/\emptyset.$$

Suppose $(\bar{a}_\emptyset, \bar{b}_\emptyset) \in \theta/\emptyset \Rightarrow \bar{a}_\theta = \bar{b}_\theta \Rightarrow \bar{a}_\theta^{\sim} = \bar{b}_\theta^{\sim}, \bar{a}_{\theta_\pi} = \bar{b}_{\theta_\pi}$

$$\Rightarrow \overline{a^\sim}_\theta = \overline{b^\sim}_\theta, \bar{a}_{\pi_\theta} = \bar{b}_{\pi_\theta}$$
$$\Rightarrow (\overline{a^\sim}_\emptyset, \overline{b^\sim}_\emptyset), (\bar{a}_{\pi\emptyset}, \bar{b}_{\pi\emptyset}) \in \theta/\emptyset$$
$$\Rightarrow (\bar{a}_\emptyset^{\sim}, \bar{b}_\emptyset^{\sim}), (\bar{a}_{\emptyset\pi}, \bar{b}_{\emptyset\pi}) \in \theta/\emptyset.$$

Therefore $\theta/\emptyset$ is a congruence relation on $A/\emptyset$.

Theorem 6: Suppose θ_1, θ_2 are two congruences on an A*-algebra A such that $\theta_1 \supset \emptyset, \theta_2 \supset \emptyset$. Then $\theta_1 \supset \theta_2$ if and only if $\theta_1/\emptyset \supset \theta_2/\emptyset$. In particular, $\theta_1/\emptyset =$

$\theta_2/\emptyset$ implies $\theta_1 = \theta_2$.

Proof: Suppose $\theta_1 \supset \theta_2$.

Claim: $\theta_1/\emptyset \supset \theta_2/\emptyset$.

Suppose $(\bar{a}_\emptyset, \bar{b}_\emptyset) \in \theta_2/\emptyset \Rightarrow \bar{a}_{\theta_2} = \bar{b}_{\theta_2} \Rightarrow (a,b) \in \theta_2$

$\Rightarrow (a,b) \in \theta_1$(since $\theta_1 \supset \theta_2$)

$\Rightarrow \bar{a}_{\theta_1} = \bar{b}_{\theta_1} \Rightarrow (\bar{a}_\emptyset, \bar{b}_\emptyset) \in \theta_1/\emptyset$.

Therefore $\theta_1/\emptyset \supset \theta_2/\emptyset$.

Conversely suppose $\theta_1/\emptyset \supset \theta_2/\emptyset$.

Claim: $\theta_1 \supset \theta_2$.

$(a,b) \in \theta_2 \Rightarrow \bar{a}_{\theta_2} = \bar{b}_{\theta_2} \Rightarrow \bar{a}_{\theta_1} = \bar{b}_{\theta_1}$ (since $\theta_1/\emptyset \supset \theta_2/\emptyset$)

$\Rightarrow (a,b) \in \theta_1$.

Therefore $\theta_1 \supset \theta_2$.

Therefore $\theta_1 \supset \theta_2 \Leftrightarrow \theta_1/\emptyset \supset \theta_2/\emptyset$.

Clearly, $\theta_1/\emptyset = \theta_2/\emptyset \Leftrightarrow \theta_1 = \theta_2$.

Theorem 7: Any congruence $\bar{\theta}$ on $A/\emptyset$ has the form $\theta/\emptyset$, where θ is a congruence relation on the A*-algebra A such that $\theta \supset \emptyset$.

Proof: Suppose $\bar{\theta}$ is a congruence on $A/\emptyset$.

Define $\theta = \{(a, b) \mid a, b \in A, (\bar{a}_\emptyset, \bar{b}_\emptyset) \in \bar{\theta}\}$.

Claim: θ is a congruence on A.

Clearly θ is an equivalence relation.

Suppose $(a, b), (c, d) \in \theta \Rightarrow (\bar{a}_\emptyset, \bar{b}_\emptyset), (\bar{c}_\emptyset, \bar{d}_\emptyset) \in \bar{\theta}$

$\Rightarrow (\bar{a}_\emptyset \wedge \bar{c}_\emptyset, \bar{b}_\emptyset \wedge \bar{d}_\emptyset) \in \bar{\theta}$($\because \bar{\theta}$ is a congruence)

$\Rightarrow (\overline{(a \wedge c)}_\emptyset, \overline{(b \wedge d)}_\emptyset) \in \bar{\theta}$

$\Rightarrow (a \wedge c, b \wedge d) \in \theta$.

Suppose $(a, b), (c, d) \in \theta \Rightarrow (\bar{a}_\emptyset, \bar{b}_\emptyset), (\bar{c}_\emptyset, \bar{d}_\emptyset) \in \bar{\theta}$

$\Rightarrow (\bar{a}_\emptyset * \bar{c}_\emptyset, \bar{b}_\emptyset * \bar{d}_\emptyset) \in \bar{\theta}$ ($\because \bar{\theta}$ is congruence)

$\Rightarrow (\overline{(a * c)}_\emptyset, \overline{(b * d)}_\emptyset) \in \bar{\theta}$

$\Rightarrow (a * c, b * d) \in \theta$.

Suppose $(a, b) \in \theta \Rightarrow (\bar{a}_\emptyset, \bar{b}_\emptyset) \in \bar{\theta}$

$\Rightarrow (\overline{a_\emptyset}^{\sim}, \overline{b_\emptyset}^{\sim}) \in \bar{\theta}$ ($\because \bar{\theta}$ is a congruence)

$\Rightarrow (\overline{a^{\sim}}_\emptyset, \overline{b^{\sim}}_\emptyset) \in \bar{\theta}$

$\Rightarrow (a^{\sim}, b^{\sim}) \in \theta$.

Suppose $(a, b) \in \theta \Rightarrow (\bar{a}_\emptyset, \bar{b}_\emptyset) \in \bar{\theta}$

$\Rightarrow (\bar{a}_{\emptyset\pi}, \bar{b}_{\emptyset\pi}) \in \bar{\theta}$ (since $\bar{\theta}$ is a congruence)

$\Rightarrow (\bar{a}_{\pi\emptyset}, \bar{b}_{\pi\emptyset}) \in \bar{\theta}$

$\Rightarrow (a_\pi, b_\pi) \in \theta$.

Therefore θ is a congruence relation on the A*-algebra A.

Claim: $\bar{\theta} = \theta/\emptyset$.

$(\bar{a}_{\emptyset}, \bar{b}_{\emptyset}) \in \bar{\theta} \Leftrightarrow (a, b) \in \theta \Leftrightarrow \bar{a}_{\theta} = \bar{b}_{\theta} \Leftrightarrow (\bar{a}_{\emptyset}, \bar{b}_{\emptyset}) \in \theta/\emptyset$.

Therefore $\bar{\theta} = \theta/\emptyset$.

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