ROTATION IN HIGHER DIMENSION BY USING EXTENDED CROSS PRODUCT

Joo Won Park[1], Byeong Hak Lee[2]

Abstract: ***Rotation in mathematics has been interesting field of study of which the concept is learned from early education. This paper attempts to develop a function that will calculate the result of a rotation in higher dimensions. It reviews three-dimensional rotation that involves cross product and demonstrate the extension of rotation to higher dimensions.***

Keywords: Cross product, dimension, rotation, vector

1. MOTIVATION

The earliest attempt of rotation was as simple as drawing the subject on a paper and actually rotating the paper itself. The advanced mathematics yet suggests techniques of analytical geometry or linear algebra for rotation.

The concept of higher dimension is not understood clear-cut, yet the approach of extending three-dimensional procedures into higher dimensions is very interesting and applicable for other concepts.

This paper combines our interest and curiosity in rotation and in higher dimensions and examines a method to define mathematical rotation in higher dimension. It attempts to develop a function for rotation in higher dimension by using extended cross product, starting from fourth and then generalizing to the nth dimension.

2. EXTENDED CROSS PRODUCT

Conventional cross product is only defined in three-dimensional space. Therefore, this paper newly defines cross product in high dimensional space, namely 'extended cross product'.

1) Cross product in third dimension

In third dimension, the cross product is defined as following:

Given the vectors $\vec{a}$, $\vec{b}$ *in three-dimensional space and angle* θ *(the smallest angle between the two vectors), the magnitude of cross product* $\|\vec{a}\times\vec{b}\|$ *is* $\|\vec{a}\|\,\|\vec{b}\|\sin\theta$ *and the direction of* $\vec{a}\times\vec{b}$ *is determined by 'right hand rule'.*

There are also few important properties of cross product that would further be applied throughout the paper:

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- $\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$.
- The cross product can be expressed as the determinant of a matrix:

If $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$, then

$$\vec{a} \times \vec{b} = \det \begin{pmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

(Note: $\hat{e}_k(k = 1, 2, \cdots n)$ is normal unit vector in n-dimensional space)

2) Cross product in higher dimensions

To apply three-dimensional cross product's form and properties, extended cross product in n-dimensional space ($n \geq 3$) is defined by matrix determinant as following:

$\forall i \in N$ s.t $i \leq n-1$, let $\vec{v}_i = (v_{i1} v_{i2}, \ldots, v_{in}) = \sum_{k=1}^{n} v_{ik} \hat{e}_k$. Also, let extended cross product operator in n-dimensional space C_n.

$$\text{Then, } C_n(\vec{v}_1, \ldots, \vec{v}_{n-1}) = \det \begin{pmatrix} \hat{e}_1 & \cdots & \hat{e}_n \\ v_{11} & \cdots & v_{1n} \\ \vdots & \ddots & \vdots \\ v_{(n-1)1} & \cdots & v_{(n-1)n} \end{pmatrix},$$

which is similar to the case in three dimension.

3. ROTATION IN THIRD DIMENSION USING CROSS PRODUCT

Here introduces a method that computes the rotation in three-dimensional space by using vectors: this is what this paper will further explore and generalize from third dimensional space to higher dimensions.

When an axis unit vector $\hat{n}$ and the rotation angle θ that rotates the vector $\vec{v}$ at a plane perpendicular to the axis are given, then the rotated vector $\vec{v}'$ can be expressed through cross product.

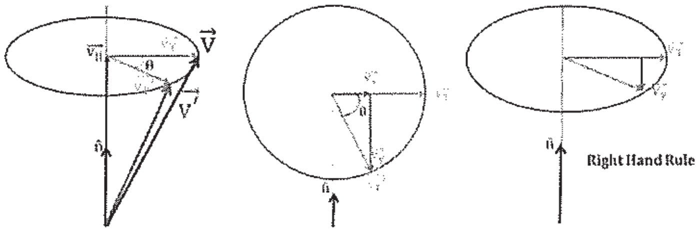

Fig. 1

The following procedure is shown in Fig. 1. The general idea is to divide the vector $\vec{v}$ into two parts: the part that does not change during rotation ($\vec{v}_{\|}$) and the part that does change ($\vec{v}_T$): $\vec{v} = \vec{v}_{\|} + \vec{v}_T = (\vec{v}\cdot\hat{n})\hat{n} + \vec{v}_T$. The rotated vector $\vec{v}\,'$ can be divided likewise into $\vec{v}_{\|}$ and $\vec{v}_T\,'$: $\vec{v}\,' = \vec{v}_{\|} + \vec{v}_T\,'$. The magnitude of vector $\vec{v}_T\,'$ is equivalent to that of vector $\vec{v}_T$, so $\| \vec{v}_T\,' \| = \| \vec{v}_T \| = v_T$. Using this fact, $\vec{v}_T\,'$ of the rotated vector $\vec{v}\,'$ can be computed as following:

$$\vec{v}_x = \cos\theta\vec{v}_T = \cos\theta\vec{v} - \cos\theta(\vec{v}\cdot\hat{n})\hat{n}$$
$$\vec{v}_y = \sin\theta(\hat{n}\times\vec{v}_T)$$
$$\vec{v}_T\,' = \vec{v}_x + \vec{v}_y = \cos\theta\vec{v} - \cos\theta(\vec{v}\cdot\hat{n})\hat{n} + \sin\theta(\hat{n}\times\vec{v}_T)$$

Since $\vec{v}_{\|}$ of $\vec{v}\,'$ is not changed, $\vec{v}\,'$ can be expressed as $\vec{v}\,' = \cos\theta\vec{v} + (1-\cos\theta)(\vec{v}\cdot\hat{n})\hat{n} + \sin\theta(\hat{n}\times\vec{v}_T)$.

This method of rotation can be simply expressed as $R(\hat{n}, \theta, \vec{v})$, which R refers to a rotation function for a given vector $\vec{v}$.

4. ROTATION IN FOURTH DIMENSION USING EXTENDED CROSS PRODUCT:

The third dimensional rotation needed a fixed first dimensional axis, which is a line, and a plane perpendicular to that axis that could rotate. Likewise, the fourth dimensional rotation needs a second dimensional axis, which is noted as "axial plane" in this paper, and a plane perpendicular to the plane. The rotation in fourth dimension involves two angles: rotation angle at the axial plane and another angle at the plane perpendicular to the axial plane. Thus, rotational function R in fourth dimension is $R(\vec{u}, \vec{v}, \theta, \phi, \vec{x})$: $\vec{u}$ and $\vec{v}$ are vectors that form the axial plane ($\vec{u} \perp \vec{v}$), θ and ϕ are rotational angles, and $\vec{x}$ is the given vector.

Note that the direction of cross product in fourth dimension cannot be determined by right-hand rule as in three-dimensional space: the direction can be defined by extended cross product as the direction in three dimension is defined by cross product.

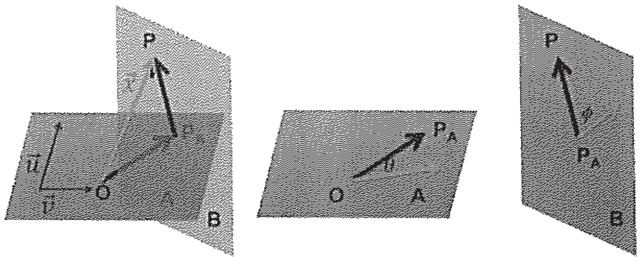

Fig. 2

Let $\overrightarrow{OP} = \vec{x}, \overrightarrow{OP_A} = \vec{x}_A, \overrightarrow{P_A P_B} = \overrightarrow{x_B}$. Plane A is the axial plane formed by two orthogonal vectors $\vec{u}$ and $\vec{v}$, and plane B is a plane orthogonal to A that contains P_A. Fig. 2 shows the rotation procedure in fourth dimension. The vector is divided in two components so that each is on planes A and B ($\overrightarrow{OP_A} = \overrightarrow{x_A}$ and $\overrightarrow{P_A P_B} = \overrightarrow{x_B}$). The rotation takes place in both planes by angles θ and ϕ. The following will elaborate how to compute $\vec{x}'$ with the idea of extended cross product.

Vector $\vec{x}$'s component parallel to plane A can be expressed as $\overrightarrow{x_A} = (\vec{x} \cdot \vec{u})\vec{u} + (\vec{x} \cdot \vec{v})\vec{v}$. This vector is rotated by angle θ. Using matrix algebra, $\overrightarrow{x_A}'$resulted after the rotation can be easily expressed as

$$\vec{x}_A' = ((\vec{x} \cdot \vec{u})\vec{u} + (\vec{x} \cdot \vec{v})\vec{v})' = \begin{pmatrix} \vec{u} & \vec{v} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \vec{x} \cdot \vec{u} \\ \vec{x} \cdot \vec{v} \end{pmatrix}$$

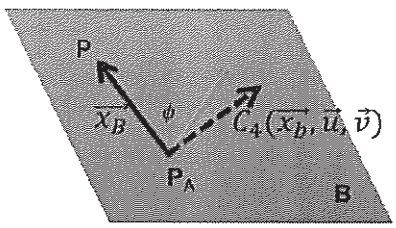

Fig. 3

The rotation of $\overrightarrow{x_B}$, vector $\vec{x}$'s component parallel to plane B, can be expressed by using extended cross product. The result of rotation by angle ϕ, $\overrightarrow{x_B}'$, is linear combination of $\overrightarrow{x_B}$ and $C_4(\overrightarrow{x_B}, \vec{u}, \vec{v})$, which is the dotted vector in Fig. 3 that is perpendicular to $\overrightarrow{x_B}, \vec{u}$, and $\vec{v}$ that has the magnitude of $\| \overrightarrow{x_B} \| = x_B$:

$$\overrightarrow{x_B}' = \cos\phi \overrightarrow{x_B} + \sin\phi C_4(\overrightarrow{x_B}, \vec{u}, \vec{v})$$

Therefore, rotation in fourth dimension $R(\vec{u}, \vec{v}, \theta, \phi, \vec{x})$ results in

$$\vec{x}' = \vec{x}_A' + \overrightarrow{x_B}' = \begin{pmatrix} \vec{u} & \vec{v} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \vec{x}\cdot\vec{u} \\ \vec{x}\cdot\vec{v} \end{pmatrix}$$

$$+\cos\phi(\vec{x} - ((\vec{x}\cdot\vec{u})\vec{u} + (\vec{x}\cdot\vec{v})\vec{v})) + \sin\phi C_4(\overrightarrow{x_B}, \vec{u}, \vec{v})$$

5. GENERALIZATION: ROTATION IN HIGHER DIMENSIONS

The previous procedures can predict rotation in higher dimensions. In n-dimensional space (n is a natural number), we need 'axial $n-2$ dimensional figure' (as 'axial plane' in four-dimensional rotation) and $[n/2]$(greatest integer less or equal to $n/2$) rotational angles.

The rotational function in n-dimensional space R_n should be examined by two cases: 1. n is an even number ($2|n$): 2. n is an odd number ($2|n-1$).

1) $2|n$: $\overrightarrow{v_1}, \overrightarrow{v_2}, \cdots, \overrightarrow{v_{n-2}}$ form 'axial $n-2$ dimensional figure, just as $\vec{u}$ and $\vec{v}$ formed axial plane in four-dimensional rotation

$$R_n(\overrightarrow{v_1}, \overrightarrow{v_2}, \cdots, \overrightarrow{v_{n-2}}, \theta_1, \theta_2, \cdots, \theta_{n/2}, \vec{x})$$

$$= \sum_{k=1}^{(n/2)-1} \begin{pmatrix} \overrightarrow{v_{2k-1}} & \overrightarrow{v_{2k}} \end{pmatrix} \begin{pmatrix} \cos\theta_k & -\sin\theta_k \\ \sin\theta_k & \cos\theta_k \end{pmatrix} \begin{pmatrix} \vec{x}\cdot\overrightarrow{v_{2k-1}} \\ \vec{x}\cdot\overrightarrow{v_{2k}} \end{pmatrix}$$

$$+\cos\theta_{n/2}(\vec{x} - \sum_{k=1}^{n-2}((\vec{x}\cdot\overrightarrow{v_k})\overrightarrow{v_k}) + \sin\theta_{n/2} C_n(\vec{x}, \overrightarrow{v_1}, \overrightarrow{v_2}, \cdots, \overrightarrow{v_{n-2}})$$

2) $2|n-1$: $\overrightarrow{v_1}, \overrightarrow{v_2}, \cdots, \overrightarrow{v_{n-2}}$ form 'axial $n-2$ dimensional figure, just as $\vec{u}$ and $\vec{v}$ formed axial plane in four-dimensional rotation

$$R_n(\overrightarrow{v_1}, \overrightarrow{v_2}, \cdots, \overrightarrow{v_{n-2}}, \theta_1, \theta_2, \cdots, \theta_{(n-1)/2}, \vec{x})$$

$$= (\vec{x}\cdot\vec{v_1})\vec{v_1} + \sum_{k=1}^{(n-3)/2}\begin{pmatrix} \overline{v_{2k}} & \overline{v_{2k+1}} \end{pmatrix}\begin{pmatrix} \cos\theta_k & -\sin\theta_k \\ \sin\theta_k & \cos\theta_k \end{pmatrix}\begin{pmatrix} \vec{x}\cdot\overline{v_{2k}} \\ \vec{x}\cdot\overline{v_{2k+1}} \end{pmatrix}$$

$$+\cos\theta_{(n-1)/2}(\vec{x}-\sum_{k=1}^{n-2}((\vec{x}\cdot\vec{v_k})\vec{v_k}) + \sin\theta_{(n-1)/2}C_n(\vec{x},\vec{v_1},\vec{v_2},\cdots,\overline{v_{n-2}})$$

6. CONCLUSION

So far this paper has studied mathematical rotation in higher dimension and constructed a formula that allows such procedure by applying extended cross product. The abstract idea of cross product in fourth dimension is mathematically defined and generalized to n-dimension by extending the three-dimensional procedures.

We look forward to further researching the transformation in higher dimensions and the meanings: whether other mathematics application in three-dimensional space can be extended, or studying the meaning of higher dimensional procedures not limited to mathematical formulas.

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8. REFERENCES

1. Article: "Euler, Look At (UDFR), Quaternion", Naver blog, http://blog.naver.com/wh0aml?Redirect=Log&logNo=80159307367
2. Article: "Rotation (mathematics)", Wikipedia, http://en.wikipedia.org/wiki/ Rotation_(mathematics)
3. Article: "Looking at the nubers: 5. Quaternion", tistory blog, http://tetration.tistory.com/23
4. Article: "Application of quaternions", Naver cast, http://navercast.naver.com/contents.nhn?contents_id=5036&path=|453|490|&leafId=644
5. Article: "Euler angles", Wikipedia, http://en.wikipedia.org/wiki/Euler_angles
6. Article: "Euler Angles", Wolfram Math World, http://mathworld.wolfram.com/EulerAngles.html

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[1]*Joo Won Park: Seocho-gu Bangbae-dong Sinsamho apt. Ba-104, Seoul, Republic of Korea /Korean Minjok Leadership Academy/ jwprk555@gmail.com*
[2]*Byeong Hak Lee: Manan-gu Anyang3-dong 695-126, Anyang, Gyeonggi, Republic of Korea /Korean Minjok Leadership Academy/ bluelbh0307@hanmail.net*