SHEAF REPRESENTATION OF P-NEAR RINGS

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***Abstracts:** In Mathematics, a sheaf is a tool for systematically tracking locally defined data attached to the open sets of a topological space. Sheaves exist in several varieties such as sheaves of sets, sheaves of rings, etc. Sheaves have several applications in topology and especially in algebraic and differential geometry. Grothendick [04] proved that a commutative ring is isomorphic with ring of sections of a sheaf of local rings. George Szeto [03] proved that a near –ring with identity in which every element is a power of itself is isomorphic with a near-ring of sections of a sheaf of near-fields in which every element is a power of itself. In this paper we prove that a p-near-ring with identity is isomorphic with a near- ring of a sheaf of p- near- rings with identity.*

***Keywords:** topological space, sheaves of sets, sheaves of rings, sheaf of p- near- rings*

Introduction

1.1 Definition: Let X be a topological space. A sheaf F of sets on X if it satisfies :

I. (a) For each open set U of X, a set F(U) (called the set of sections of F over U).

(b) For each pair of open sets $V \subseteq U$ of X, a restriction map $\rho_v^u : F(U) \rightarrow F(V)$ such that

(i) for all U, $\rho_u^u = id_u$.

(ii) whenever $W \subseteq V \subseteq U$ (all open) $\rho_w^u = \rho_w^v \circ \rho_v^u$.

II. If $U = \bigcup_{\lambda \in \Lambda} U_\lambda$ is open covering of U and s, $s^I \in F(U)\ \forall\ \lambda \in \Lambda$,

$\rho_{u_\lambda}^u (s) = \rho_{u_\lambda}^u (s^I)$ then $s = s^I_\lambda$.

III. If U is open in x, $U = \bigcup_{\lambda \in \Lambda} U_\lambda$ is an open covering of U and $(S_\lambda)_{\lambda \in \Lambda}$,

is a family of sections of F with $\forall\ \lambda \in \Lambda$, $s_\lambda \in F(U_\lambda)$, such that

$\forall\ \lambda, u$, $\rho^{u_\lambda}_{u_\lambda \cap u_u} (s_\lambda) = \rho^{u_u}_{u_\lambda \cap u_u} (s_u)$, then there is $s \in F(U)$ such that λ

$\rho_{u_\lambda}^u (s) = s_\lambda$.

1.2 Note: If F satisfies only I, then F is called presheaf of sets over X.

1.3 **Definition:** Suppose F is a presheaf of sets over a topological space and $x \in X$. $B_x = \{ U / U$ is open set in X and $x \in U\}$. Then $F_x = \underset{U \in Bx}{\underline{\text{limit}}} F(U)$.

is called stalk of F at x ; this comes equipped with maps $F(U) \to Fx$: $s \to s_x$ whenever an open set $U \ni x$, the members of F_x are called germs.

1.4 **Theorem:** (a) Each germ $t \in F_x$ arises as $t = s_x$ for some $s \in F(U)$ for some open neighborhood U of x.

(b)Two germs $s_x, t_x \in F_x$ (with $s \in F(U)$, $t \in F(v)$ say) are equal i.e. $s_x = t_x \Leftrightarrow$ $\exists$ open set $W \subseteq U \cap V$ such that $\rho_w{}^u(s) = \rho_u{}^v(t)$.

1.5 **Note** : (a) IF F, G are presheaves of 3-rings, for f to be a morphism of presheaves of 3-rings we require each f(U) to be a homomophism of 3-rings.

(b) Suppose F,G,H are presheaves over X and $f : F \to G$, $g : G \to H$ are presheaf homomophism then $gof : F \to H$ is a presheaf homomophism where $(gof)(U)=g(U) \text{ o } f(U)$. We say $f : F \to G$ is an isomorphism of presheaves (of sets or of 3-rings) iff there is a morphism $g:G \to F$ such that $fog = id_G$, $gof = id_F$ where $id_F:F \to F$ is defined by $id_F(U)= id_{F(U)}$ 6for each open set U in X.

1.6 **Theorem:** $f : F \to G$ is an isomorphism of presheaves iff $\forall$ open U of X, f(U) is an isomorphism iff $\forall$ open U of X, f(U) is bijection.

1.7 **Theorem:** F, G are presheaves over X and $f : F \to G$ is presheaves homomorphism on X, then for every $x \in X$, there is a morphism of

stalks f_x: $F_x \to G_x$ is such a way that whenever $F \xrightarrow{f} G \xrightarrow{g} H$ we have

$(gof)_x = g_x \text{ o } f_x$

1.8 **Proposition:** A presheaf F is a sheaf iff whenever $U = \cup U_\lambda$ is an open cover of an open set U, the associated diagram (IV of 1.62 Note) of sets is an equalizer diagram i.e, a is an equalizer of (b,c).

1.9 **Theorem:** If F is a presheaf and G is a monopresheaf over X, and f, g : $F \to G$ are two morphisms such that$\forall x \in X$, $f_x = g_x$ (i.e, f,g agree on all stalks) then $f = g$.

1.10 **Example:** Let E be a topological space and $\rho : E \to X$ a continuous map. We can construct sheaf F of sections of ρ : for U open in X, let $F(U) = \{ \sigma / \sigma : U \to E$ is continuous and $\rho \text{ o } \sigma = id_U$ i.e,

For open $V \subseteq U$, $\rho_v{}^u : F(U) \to F(V)$ by $\rho_v{}^u(\sigma) = \sigma / V$.

1.11 **Theorem:** If X is a topological space and F a sheaf on X, then for any open U and $s, s' \in F(U)$ we have $s = s' \Leftrightarrow \forall x \in U$, $s_x = s'_x$.

1.12 **Definition:**let X be A topological space . A sheaf space over X is a pair (E, P) of a topological space E and a continuous map $\rho : E \to X$ such that ρ is a local homeomorphism i.e $\forall\ y \in E, \exists$ open $N \subseteq E \ni y \in N$ and open $U \subseteq X \ni \rho(y) \in U \ni \rho/N : N \to U$ is homeomorphism.

1.13 **Theorem**: For each sheaf space E, there is a sheaf of sets ΓE in such a way that a morphism $f : E \to E'$ of sheaf spaces gives rise to a morphism $\Gamma f : \Gamma E \to \Gamma E'$ of sheaves.

1.14 **Proposition**: If (E, ρ) is a sheaf space, then the stalk of ΓE at $x \in X$, is (upto isomorphism) just the fibre $\rho^{-1}(x)$ of ρ over x, which has the discrete topology as a subspace of E.

1.15 **Note:** (i) $\Gamma(f \circ g) = \Gamma f \circ \Gamma g$ and $\Gamma(\text{id}) = \text{id}$

(2) $(\Gamma f)_x : (\Gamma E)_x \to (\Gamma E')_x$

$f/\rho^{-1}(x) : \rho^{-1}(x) \to \rho'^{-1}(x)$ are isomorphic maps.

1.16 **Theorem** : For each presheaf F on X there is a sheaf space in such a way that any morphism $f : F \to F'$ of presheaves gives rise to $Lf : Lf \to Lf'$ of sheaf spaces.

1.17 **Theorem** : If E is a sheaf space over X, then $L\Gamma E$ is isomorphic to E as sheaf spaces over X (i.e. there is a morphism $\phi : E \to L\Gamma E$ of sheaf spaces with a two sided inverse).

1.18 **Definition**: F is a presheaf over X, then we obtain a sheaf $\Gamma L F$ n_F $: F \to \Gamma L F$ by $n_F(U) : \hat{s} \hookrightarrow s$ where U is open in X, $x \in U$, $s \in F(U)$, $s : U \to L F$ by $s(x) = s_x$.

1.19 **Theorem** : Let F be a presheaf and G a sheaf over X. For any $f : F \to G$ morphism of presheaves, there is a unique sheaf morphism

$g : \Gamma L F \to G$ such that

Commutes

1.20 **Lemma**: If (and only if) G is a sheaf, then (and only then) $G \to \Gamma L G$ is an isomorphism of sheaves.

1.21 **Definition:** A directed set Λ is a set with a pre-order $\leq$ (i.e a reflexive and transitive relation : $\alpha \leq \alpha$ and $\alpha \leq \beta \leq \gamma \Rightarrow \alpha \leq \gamma$) which also satisfies:

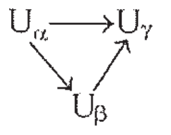
(a) $\forall\ \alpha, \beta \in \Lambda\ \exists\ \gamma \in \Lambda$ such that $\alpha \leq \gamma$ and $\beta \leq \gamma$.

We often write $\Lambda_1 = \{ (\alpha, \beta) \in \Lambda \times \Lambda ; \alpha \leq \beta \}$.

A direct system of sets indexed by a directed set Λ is a family $(U_\alpha)_{\alpha \in \Lambda}$ of sets together with, for each $(\alpha, \beta) \in \Lambda_1$, a map of sets $\rho_{\alpha\beta} : U_\alpha \to U_\beta$, satisfying

(b) $\forall\ \alpha \in \Lambda$, $\rho_{\alpha\alpha} = \text{id}_{U\alpha}$.

(c) $\forall\ \alpha, \beta, \gamma \in \Lambda$ if $\alpha \leq \beta \leq \gamma$ then the triangle



commutes, i.e $\rho_{\alpha\gamma} = \rho_{\beta\gamma} \circ \rho_{\alpha\beta}$.

1.22 **Example:** Given a topological space X, the set T of its open sets is directed by the relation $U \leq V \Leftrightarrow U \supseteq V$ (condition (a) holds since U $\cap$ V is open if U , v are).

Main Section

2.1 **Definition:** Let X be a topological space. Suppose that for $x \in X$, a ring R_x with zero 0_x and identity 1_x. Assume $R_x \cap R_y = \varnothing$ if $x \neq y$. Let R $= \cup R_x$. Denote $\Pi : R \rightarrow X$ by $\Pi(r) = x$ if $r \in R_x$. Assume that a topology is imposed on R such that the following axioms are satisfied :

I. If $r \in R$, $\exists$ open sets U in R with $r \in U$, $N \subseteq X$ such that $\Pi : U \rightarrow N$ is homeomorphic.

II. Let $R + R = \{(r,s) / \Pi(r) = \Pi(s)\}$ with the topology induced by the product topology in $R \times R$. Then the mapping $r \rightarrow -r$ is continuous on R to R and the mappings $(r, s) \rightarrow r + s$, $(r, s) \rightarrow rs$ are continuous on R+R to R.

III. The mapping $x \rightarrow 1_x$ is continuous on X to R.

With these conditions R is called a sheaf of rings over X. The rings R_x are called the stalks of the sheaf R. The pair (X, R) is called a ringed space.

2.2 **Definition:** Suppose R is a sheaf of rings. A subset of R^1 of R is a subsheaf of R if R^1 is an open subset of R and $R^1 \cap R_x$ is a subring of R_x for each $x \in X$. Then R^1 is a sheaf of rings over X.

2.3 **Definition:** Let R be a sheaf of rings over X. Let U be a subset of X. A section of R over U is a continuous mapping σ of U to R such that $\Pi \circ \sigma = id_u$. The collection of sections of R over U is denoted $\Gamma(U)$.

2.4 **Lemma:** Let R be a sheaf of rings over a space X. (a). The mapping $\Pi : R \rightarrow X$ is open and continuous. (b). $\Gamma(U)$ is a ring with pointwise operations $(\sigma_1 + \sigma_2)(x) = \sigma_1(x) + \sigma_2(x)$, $(\sigma_1\sigma_2)(x) = \sigma_1(x)\,\sigma_2(x)$, $0(x) = 0_x$, and $1(x) = 1_x$. (c). Let $r \in R_x$, then $\exists$ a nbd N of X and $\sigma \in \Gamma(N)$ such that $\sigma(x) = r$. (d). Let $x \in U \subseteq X$ and suppose that $\sigma_1 \in \Gamma(U)$, $\sigma_2 \in \Gamma(U)$ satisfy $\sigma_1(x) = \sigma_2(x)$. Then $\exists$ a nbd N of X such that $\sigma_1(y) = \sigma_2(y)$ for all $y \in N \cap Y$.

Proof: (a). $\Pi : R \rightarrow X$. Let N_1 be an open set.

Let $r \in \Pi^{-1}(N_1) \Rightarrow \Pi(r) \in N$. Since $r \in R$ $\exists$ an open set U of R and open set N_2 is homeo and $U \cap \Pi^{-1}(N_2) \subseteq \Pi^{-1}(N_1)$. $\therefore$ Π is continuous.

Π is open map : Let W be an open set in R and $x \in \Pi(W)$. Let $e \in W \ni \Pi(e) = x$. Since $e \in R$, so $\exists$ a nbd $W^1 \subseteq W$ & $e \in W^1$ mapped by Π onto an open set in X i.e x has a nbd $\Pi(W^1)$ inside $\Pi(W)$.

$\therefore$ Π is open map.

(b). Let $\sigma_1, \sigma_2 \in \Gamma(U) \Rightarrow \sigma_1 , \sigma_2 : U \to R$

$(\sigma_1 + \sigma_2)(x) = \sigma_1(x) + \sigma_2(x)$ $\quad$ $(\sigma_1\sigma_2)(x) = \sigma_1(x)\,\sigma_2(x)$

$0(x) = 0_x , 1(x) = 1_x$ $\quad$ $\forall x \in U$.

$\sigma_1 + \sigma_2 : U \to R$

Let W be a nbd in R.

Consider $(\sigma_1 + \sigma_2)^{-1}(W)$

Let $x \in (\sigma_1 + \sigma_2)^{-1}(W)$ $\quad \Rightarrow (\sigma_1 + \sigma_2)(x) \in W$

$\Rightarrow \sigma_1(x) + \sigma_2(x) \in W$

Since + is continuous $\exists$ nbd W^1 of $\sigma_1(x) + \sigma_2(x) \ni W^1 \subseteq W$.

Claim: $\sigma_1 + \sigma_2$ is continuous.

Let W open in R. Let $x \in (\sigma_1 + \sigma_2)^{-1}(W) \Rightarrow \sigma_1(x) + \sigma_2(x) \in W$.

Since $+ : R \times R \to R$ is continuous and W nbd in R

$\Rightarrow +^{-1}(W)$ is open in $R \times R$. $\quad$ Since $(\sigma_1(x) , \sigma_2(x)) \in +^{-1}(W)$

$\exists$ W_1 nbd of $\sigma_1(x)$, W_2 nbd of $\sigma_2(x) \ni W_1 \times W_2 \in +^{-1}(W)$

$\Rightarrow + (W_1 \times W_2) \subseteq W$ and $W_1 + W_2$ is nbd of $\sigma_1(x) + \sigma_2(x)$.

$\therefore$ $\sigma_1 + \sigma_2$ is continuous.

111^{ly} $\sigma_1\sigma_2$, $0(x) = 0_x$, $1(x) = 1_x$ are continuous.

(c). Let $r \in R_x \Rightarrow r \in R \Rightarrow \exists$ open sets U in R with $r \in U$, and $N \subseteq X$.

$\Pi : U \to N$ is homeomorphism.

Let $\sigma = (\Pi/U)^{-1}: N \to U \subseteq R \Rightarrow \sigma \in \Gamma(N)$.

Since $\Pi(r) = x \Rightarrow (\Pi/U)^{-1}(x) = r$

$\Rightarrow \sigma(x) = r$.

(d). Suppose N is open & $\sigma \in \Gamma(N)$.

$\Rightarrow \Pi \circ \sigma = id_N$

$\Rightarrow \sigma$ is one-one and $\sigma^{-1} = \Pi/\sigma(N)$.

Let $x \in N$ and choose a nbd U of $\sigma(x)$ in R such that the maps U homeomorphiccally on a nbd of x in X.

But the continuity of σ, $\sigma^{-1}(U \cap \sigma(N)) = \sigma^{-1}(U)$ is nbd M of x. Then $\sigma(M) = (\Pi/U)^{-1}(M)$ is open.

Since x is arbitrary, $\sigma(N)$ is open.

$\therefore 0(x) = \{ 0_x / x \in X \}$ is open.

(e). Let $x \in U \subseteq X$ and $\sigma_1 \in \Gamma(U)$, $\sigma_2 \in \Gamma(U) \ni \sigma_1(x) = \sigma_2(x)$

Since $\Pi\sigma_1(x) = \Pi\sigma_2(x)$

$\Rightarrow (\sigma_1(x), \sigma_2(x)) \in R + R$ & + is continuous on R + R $\exists$ nbd N of x $\ni \sigma_1(y) = \sigma_2(y)\ \forall\ y \in N \cap U$.

2.5 **Definition:** Partition property of a Boolean space: If X is a Boolean space (totally disconnected compact hausdorff space), and $\{N_i / i \in I\}$ is an open covering of X, then $\exists\ \{M_1, M_2, \ldots\ldots\ldots, M_r\}$, a finite set of clopen subsets of X, such that

(i) for every $j \leq r$, $\exists\ i \in I \ni M_j \subseteq N_i$.

(ii) $M_i \cap M_j = \phi$ if $i \neq j$

(iii) $\bigcup_{i=1}^{r} M_i = X$

The Collection $\{M_1, M_2, \ldots\ldots\ldots, M_r\}$ of open closed subsets is called a partition of X.

2.6 **Note:** (1) If $\sigma \in \Gamma(U)$ & $V \subseteq U$, then $\sigma/V \in \Gamma(V)$.

(2). If U_1, U_2 are subsets of X with $\bar{U}_1 \cap \bar{U}_2 = \phi$, and if $\sigma_i \in \Gamma(U_i)$, then

$\sigma_1 \vee \sigma_2 \in \Gamma(U_1 \cup U_2)$ when $(\sigma_1 \vee \sigma_2)(y) = \sigma_1(y)$ if $y \in U_1$

$= \sigma_2(y)$ if $y \in U_2$.

2.7 **Theorem:** Let R be a sheaf of rings over a Boolean space X. Let U be a closed subset of X and suppose that $\sigma \in \Gamma(U)$. Then there exists $\tau \in \Gamma(X)$ such that $\tau / U = \sigma$.

Proof: Let $x \in X$. Suppose $x \notin U$, $\exists$ nbd N_x of $x \ni U \cap N_x = \phi$.

$\therefore X = \bigcup_{x \in X} N_x$.

By the partition property $\exists$ a finite set $\{M_1, M_2, \ldots\ldots\ldots, M_r\}$ of clopen sets in X $\ni$ (i). $M_j \cap M_k = \phi$ for $j \neq k$.

(iii) $\bigcup_{i=1}^{r} M_i = X$, and for each j, either $M_j \cap Y = \phi$ or $\exists\ \tau_j \in \Gamma(M_j) \ni$

$\tau_j(y) = \sigma(y)\ \forall\ y \in M_j \cap Y$. If $M_j \cap Y = \phi$, let $\tau_j(y) = 0_y$ for all $y \in M_j$
Let $\tau = \tau_1 \cup \tau_2 \cup \ldots\ldots \cup \tau_r$. Then $\tau \in \Gamma(X)$ and $\tau / U = \sigma$.

2.8 **Note:** X is a Boolean space. R is a sheaf of rings. The mapping from $\Gamma(X, R) \rightarrow R_x$ is an epimorphism $\sigma \rightarrow \sigma(x)$. Let $r \in R_x \Rightarrow r \in R$.

Let $U = \{x\} \subseteq X \Rightarrow U$ is closed set in X & put $\sigma^1(z) = r$.

$\Rightarrow \exists\ \sigma \in \Gamma(x) \ni \sigma/U = \sigma^1$ i.e $\sigma(x) = \sigma^1(x) = r$.

$|\sigma| \rightarrow \sigma(x)$ is an epi..

2.9 **Definition:** A ringed space (X, R) will be called reduced if

(i) X is Boolean space.

(ii) If $\sigma \in B(\Sigma(X))$, then for all $x \in X$, either $\sigma(x) = 0_x$ or $\sigma(x) = 1_x$.

2.10 **Theorem:** Suppose (X, R) is a ringed space with X a Boolean space if R_x is commutative $\forall\ x \in X$ then $\Gamma(X)$ is commutative.

Proof: Suppose is commutative $\forall\ x \in X$.

Let $\sigma_1, \sigma_2 \in \Gamma(X)$.

$\sigma_1(x) + \sigma_2(x) - \sigma_1(x) - \sigma_2(x) = 0\ \exists$ nbd M_x of $x \ni$

$\sigma_1(y) + \sigma_2(y) - \sigma_1(y) - \sigma_2(y) = 0\ \forall\ y \in M_x$.

Since $X = \bigcup_{x \in X} M_x$

Since X is Boolean space, by finite partition of X, $\exists$ a finite family $\{M_1, M_2, \ldots\ldots\ldots, M_r\}$ of pair wise disjoint, clopen sets with

$\sigma_1(y) + \sigma_2(y) - \sigma_1(y) - \sigma_2(y) = 0\ \forall\ y \in M_y$.

$\therefore\ \sigma_1(x) + \sigma_2(x) - \sigma_1(x) - \sigma_2(x) = 0\ \forall\ x \in X$.

$\therefore\ \sigma_1(x) + \sigma_2(x) = \sigma_2(x) + \sigma_1(x)\ \forall\ x \in X$.

$\therefore\ \sigma_1 + \sigma_2 = \sigma_2 + \sigma_1$

$\therefore\ \Gamma(X)$ is commutative.

2.11 **Lemma :** Let (X, R) be commutative ringed space with X is a Boolean space. Then

(1). If for all x, R_x is indecomposable, then (X, R) is reduced.

(2). If for all x, R_x is commutative and (X, R) is reduced then $\forall$ x, R_x is indecomposable.

Proof: $B(\Gamma(X)) = \{\ \sigma / \sigma : X \rightarrow R$ is continuous and $\sigma^2 = \sigma\ \}$.

Since $\sigma \rightarrow \sigma(x)$ is an epimorphism from $\Gamma(X)$ onto R_x.

Since $\sigma^2 = \sigma \Rightarrow \sigma(x) \in B(R(x))$.

Since R_x is indecomposable, $B(R_x) = \{ 0_x, 1_x\}$.

If $\sigma \in B(\Gamma(X)) \Rightarrow \sigma(x) = 0_x$ or $\sigma(x) = 1_x$.

$\therefore$ (X, R) is reduced.

Suppose R_x is commutative $\forall$ x and (X, R) is reduced. Then $\Gamma(X)$ is commutative.

Claim: R_x is indecomposable.

Suppose some R_y is decomposable, so $\exists$ an idempotent $e_y \ni$

$e_y \neq 0_y, 1_y$.

Define $\sigma^1 : \{y\} \to R$ by $\sigma^1(y) = e_y$. Then $\exists$ a clopen set N of y $\ni$ $\tau \in \Gamma(N) \ni \tau(y) = e_y$ & $(\tau^2 - \tau)(x)(x) = 1_x \; \forall x \in N$.

Define $\sigma : X \to R$ by $\sigma(x) = \tau(x) \;\; \forall x \in N$

$$\sigma(x) = 0_x \;\; \forall x \in X - N.$$

$\Rightarrow \sigma \in B(\Gamma(X))$ & $\sigma(y) \neq 0_y$, $\sigma(y) \neq 1_y$.

$\therefore$ (X, R) is not reduced. It is a contradiction.

$\therefore$ R_x is indecomposable $\forall x \in X$.

2.12 **Remark :** Throughout this paper hereafter we assume that N is a p-near-ring (p is prime) with identity. For p-near-rings with identity we have proved in the Paper 3 the following properties.

(1) (N, +) is a commutative group.

(2) All idempotents are central.

(3) N contains a family of completely prime ideals with trivial intersection.

Already we proved that I, I^1 are ideals of N, then I I^1, $I \cap I^1$ are ideals of N, then ΣI_α is an ideal of N. The set B(N) of all idempotents of N are central. B(N) is a Boolean ring with $e \odot e^1 = ee^1$ and $e \oplus e^1 = e + e^1 - ee^1$. And the set spec B(N) of all prime ideals is a topological space with hull-kernel topology. Spec B(N) is totally disconnected compact Hausdorff space.

2.13 **Lemma:** (1). If I, I^1 are ideals of N, then I I^1, $I \cap I^1$ are ideals of N.

(2). If $\{I_\alpha / \alpha \in \Delta\}$ is a family of ideals of N, then $\sum_{\alpha \in \Delta} I_\alpha$ is an ideal of N.

Proof: Clearly $I \cap I^1$ is an ideal of N.

(1). Claim: I I^1 are ideals of N. Let $\Sigma a_i b_i$, $\Sigma a_i^1 b_i^1$ be elements of I, I^1 with a_i, $a_i^1 \in I$, $b_i, b_i^1 \in I^1$.

$\Sigma a_i b_i - \Sigma a_i^1 b_i^1 = \Sigma a_i b_i + \Sigma a_i^1(-b_i^1)$ is in I I^1, so I I^1 is group under +.

Let $n \in N$. $(\Sigma a_i b_i)\, n = \Sigma\, ai^l(-\, bi^l\, n) \in I\, I^l$

Let $z = \Sigma a_i b_i$ be an element in $I\, I^l$.

Since $z^P = z$, so $z^{P-1} \in B(N) \cap I\, I^l$.

By pierce decomposition theorem.

$$N \cong N\, z^{P-1} + N(\, z^{P-1} - 1)$$

Denote z^{P-1} by e.

Let $x, y \in N$.

Consider $[\, y(z+x) - yx](e-1)$

$[\, y(z+x) - yx\,]\, (e-1) = y(ze + x)(e - 1) - yx(e - 1)$

$= yx\, (e - 1) - yx\, (\, e - 1)$

$= 0$

$\Rightarrow\ y(z+x) - yx \in Ne$, which is in $I\, I^l$.

$\therefore$ $I\, I^l$ is an ideal of N.

(2) Claim: $\sum_{\alpha\in\Delta} I_\alpha$ is an ideal of N.

Let $\Sigma\, a_{\alpha i}$, $\Sigma\, b_{\alpha j}$ be two elements of $\Sigma\, I_\alpha$ where $a_{\alpha i} \in I_{\alpha i}$, $b_{\alpha j} \in I_{\alpha j}$.

$\Sigma\, a_{\alpha i}\ -\ \Sigma\, b_{\alpha j}\ =\ \Sigma\, a_{\alpha i} + \Sigma(-\, b_{\alpha j}\,) \in \Sigma\, I_\alpha$.

$\therefore$ $\Sigma\, I_\alpha$ is a group.

Let $n \in N$. $(\Sigma\, a_{\alpha i}\,)n = \sum_{\alpha\in\Delta} a_{\alpha i}\, n \in \Sigma\, I_\alpha$. Let $z = \sum_{\alpha\in\Delta} a_{\alpha i} \in \Sigma\, I_\alpha$.

Let $e = z^{\,P-1}$. Let $x, y \in N$ and $[\, y(z+x) - yx\,]\, (e-1) = 0$

$\Rightarrow\ y(z+x) - yx \in Ne \subseteq \sum_{\alpha\in\Delta} I_\alpha$.

$\therefore$ $\Sigma\alpha_i$ is an ideal of N.

2.14 **Theorem:** B(N) is a set of idempotents of p-near-ring N.

B(N) is a Boolean ring where $\oplus$ and O defined by

$e \oplus f = e + f - ef$ and $e\, O\, f = ef$ $\quad \forall\, e, f \in B(N)$.

2.15 **Definition:** An ideal I is called regular if $I = N\, [\, I \cap B(N)]$.

2.16 **Lemma:** Every ideal I of N is regular.

Proof: Let $a \neq 0 \in I \Rightarrow a^{P-1} = e$ is an idempotent.

$\Rightarrow$ e is central.

$\therefore$ For $n \in N$, $e_n = n^{\,P-1} \in B(N)$.

$\therefore$ $I \cap B(N) = \{ e_n / n \in I \}$.

$\therefore$ For any $n \neq 0$ in I, $n^P = n$ so $n = n \cdot e_n \in N (I \cap B(N)]$

$\therefore$ $I \subseteq N(I \cap B(N))$

Clearly $N(I \cap B(N)) \subseteq I$

For let $n \in N$ and $e \in I \cap B(N)$

Consider $n(0 + e) - n0 \in I$

$\Rightarrow ne - 0 \in I$ (since $n0 = 0$ & $N \in \eta_0$)

$\Rightarrow ne \in I$

$\therefore$ $I = N(I \cap B(N))$

$\therefore$ I is regular.

2.17 **Theorem:** Let Π be the set of all completely prime ideals of N and $\Gamma(I) = \{ p \in \Pi / I$ an ideal of N with $I \not\subseteq P \}$. Then Π is a topological space with a basic open set $\Gamma(I)$ for all ideals I of N.

Proof; Suppose I and I^1 are ideals of N.

Claim: $\Gamma(I) \cap \Gamma(I^1) = \Gamma(I I^1)$

$$\begin{aligned} P \in \Gamma(I) \cap \Gamma(I^1) &\Leftrightarrow P \in \Gamma(I) \text{ and } P \in \Gamma(I^1) \\ &\Leftrightarrow I \not\subseteq P \text{ and } I^1 \not\subseteq P \\ &\Leftrightarrow I I^1 \not\subseteq P \\ &\Leftrightarrow P \in \Gamma(I I^1) \end{aligned}$$

Suppose $\{I_\alpha / \alpha \in \Delta\}$ is a family of ideals of N

Claim: $\bigcup_{\alpha \in \Delta} \Gamma(I_\alpha) = \Gamma(\Sigma I_\alpha)$

$$\begin{aligned} P \in \bigcup_{\alpha \in \Delta} \Gamma(I_\alpha) &\Leftrightarrow P \in \Gamma(I_\alpha) && \text{for some } \alpha \in \Delta \\ &\Leftrightarrow I_\alpha \not\subseteq P && \text{for some } \alpha \in \Delta \\ &\Leftrightarrow \Sigma I \not\subseteq P \\ &\Leftrightarrow P \in \Gamma(\Sigma I_\alpha) \end{aligned}$$

$$\therefore \quad \bigcup_{\alpha \in \Delta} \Gamma(I_\alpha) = \Gamma(\Sigma_{\alpha \in \Delta} I_\alpha)$$

$\therefore$ Π is a topological space for which $\{\Gamma(I) / I$ an ideal of N $\}$ is an open base. And clearly $\Gamma(0) = \Pi$, $\Gamma(N) = \phi$.

2.18 Theorem: The topological space Π is homeomorphic with space Spec B(N) of N.

Proof: Suppose N is a p-near-ring. Then B(N) is a Boolean. Spec B(N) the set of all prime ideals of B(N) is a totally disconnected Hausdorff space. Since already proved every ideal of N is a regular ideal of N, every completely prime ideal P of N can be written as $P = N(P \cap B(N))$.

$\therefore$ For every prime ideal $x \in X$ (hereafter we denote Spec B(N) by X)

Nx is a completely prime ideal of N($Nx \in \Pi$).

Now we define $F : \Pi \to X$ by $F(P) = P \cap B(N)$.

Since P is a completely prime ideal of N, $P \cap B(N)$ is a prime ideal of B(N).

$\therefore P \cap B(N) \in X$

Suppose $F(P) = F(P^1)$

$\Rightarrow P \cap B(N) = P^1 \cap B(N)$

$a \in P \quad \Leftrightarrow a^{P-1} \in P \cap B(N)$

since a^{P-1} is an idempotent.

$\Leftrightarrow a^{P-1} \in P \cap B(N)$

$\Leftrightarrow a^{P-1} \in P^1$

$\Leftrightarrow a = a^P = a^{P-1}a \in P^1$.

$\therefore P = P^1$.

$\therefore F(P) = F(P^1) \Rightarrow P = P^1$.

$\therefore$ F is one- one.

Suppose x is a prime ideal in B(N).

Claim: Nx is a completely prime ideal of N

Clearly Nx is an ideal in N

Suppose $nn_1 \in Nx$ for any $n, n_1 \in N$.

Suppose $n \notin Nx$ and $n_1 \notin Nx$

$\Rightarrow nn^{P-1} \notin Nx$ and $n_1n_1^{P-1} \notin Nx$

$\Rightarrow n^{P-1} \notin x$ and $n_1^{P-1} \notin x$

$\Rightarrow n^{P-1}n_1^{P-1} \notin x$

$\Rightarrow n^{P-1}n_1^{P-1} \oplus x = B(N)$

$\therefore 1 = n^{P-1}n_1^{P-1} s \oplus s^1$ for some $s, s^1 \in B(N)$ with $s^1 \in X$.

$\Rightarrow N = N n^{P-1}n_1^{P-1} \oplus x$, it is a contradiction to $N n^{P-1}n_1^{P-1} \subseteq Nnn_1 \subseteq Nx$

$\therefore$ Nx is a completely prime ideal.

$\therefore$ Nx $\in \Pi$

$F(Nx) = Nx \cap B(N) = x$ in X

$\therefore$ F is onto.

Now consider $n \in N$, $\Gamma(n)$

Clearly $\Gamma(n) = \Gamma(n^{P-1})$.

$\therefore F(\Gamma(n^{P-1})) = \Gamma_0(n^{P-1})$ where $\Gamma_0(n^{P-1}) = \{x \in X / n^{P-1} \notin X\}$

But $\Gamma_0(n^{P-1})$ is a basic open set in X

$\therefore$ F is a homeomorphism.

Since X is a totally compact Hausdorff space, Π is a totally compact Hausdorff space.

2.19 Note: $x \in X \Leftrightarrow Nx \in \Pi$.

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