RINGS WITH (X, R, X) AND COMMUTATORS IN THE LEFT NUCLEUS

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***Abstract:** In this paper we consider a ring R with (x, y, x) and commutators in the left nucleus. We show that* (x, y, x) *and commutators are in the center. Using these properties, we prove that R must be a subdirect sum of a semiprime associative ring and a semiprime commutative ring.*

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1. INTRODUCTION

In [1] Albert used the identities consisting of the Jordan identity, flexible, Lie-admissible and commutators in the nucleus. His main result was that simple finite dimensional algebras are either associative algebras or Jordan. Next Kleinfeld [2] proved that semiprime rings without the Jordan identity are subdirect sums of associative and commutative rings, while prime are Commutative or associative. Then San Soucie [4] was able to drop the Lie-admissible hypothesis without losing the conclusions. More general result was obtained by Thedy [6]. In this direction Kleinfeld [3] by weakening the two remaining hypothesis of flexible and commutators in the nucleus proved the same results for semiprime and prime rings. In this section we consider a ring R with (x, y, x) and commutators in the left nucleus. We show that (x, y, x) and commutators are in the center. Using these properties, we prove that R must be a subdirect sum of a semiprime associative ring and a semiprime commutative ring.

2. PRELIMINARIES

We shall denote the commutator and the associator by $(x, y) = xy - yx$ and $(x, y, z) = (xy)z - x(yz)$ for all x, y, z in R respectively. The left nucleus N_l of a ring R is defined as $N_l = \{n \in R / (n, R, R) = 0\}$. The middle nucleus N_m of a ring R is defined as $N_m = \{n \in R/(R, n, R) = 0\}$. The right nucleus N_r of a ring R is defined as $N_r = \{n \in R/(R, R, n) = 0\}$. The nucleus N of a ring R is defined as $N = \{n \in R / (n, R, R) = (R, n, R) = (R, R, n) = 0\}$. i.e., $N = N_l \cap N_m \cap N_r$. A ring R is called prime if whenever A and B are ideals of R such that $AB = 0$, then either $A = 0$ or $B = 0$. A ring R is called semiprime if whenever A is an ideal of R, then $A^2 = 0$ implies $A = 0$.

Throughout this section we consider a ring R with (x, y, x) and commutators in the left nucleus.

i.e., $(a, b, a) \subset N_l$ (1)

and $(R, R) \subset N_l$. (2)

In every ring the following identity holds:
$(wx, y, z) - (w, xy, z) + (w, x, yz) = w(x, y, z) + (w, x, y)z$. (3)

If $S(x, y, z) = (x, y, z) + (y, z, x) + (z, x, y)$ in every ring, we have the identity $(xy, z) + (yz, x) + (zx, y) = S(x, y, z)$.

Consequently, using (2) we have
$S(x, y, z) \subset N_l$. (4)

Moreover, in every ring we have the identity
$(xy, z) = x(y, z) + (x, z)y + S(x, y, z) - (x, z, y) - (y, z, x)$. A linearization of (1) implies

$(x, z, y) + (y, z, x) \subset N_l$.
Then combining this with the above equations (4) and (2), we obtain $x(y, z) + (x, z)y \subset N_l$. (5)

Suppose that $n \in N_l$. Then with $w = n$ in (3), we obtain $(nx, y, z) = n(x, y, z)$. Combining this with (2) yields

$(nx, y, z) = n(x, y, z) = (xn, y, z)$. (6)

A combination of (5) and (6) yields
$(x(y, z) + (x, z)y, r, s) = 0$ or $(x(y,z),r,s) = - ((x,z)y,r,s)$.
This implies $(y, z)(x, r, s) = - (x, z)(y, r, s)$. (7)

Let I be the associator ideal of R. I consist of the smallest ideal which contains all associators. We note that I may be characterized as all finite sums of associators and right (or left) multiples of associators, as a consequence of (3).

We also assume that R is semiprime. Then we know that the only ideal of R which squares to zero is the zero ideal.

Main Results

Using the above properties R, we prove the following main results:

Lemma 1 : If $T = \{t \in N_l \,/\, t(R, R, R) = 0\}$, then T is an ideal of R and $T(R,R,R) = 0$.

Proof :

By substituting t for n in (6), we obtain

$(tx, y, z) = t(x, y, z) = (xt, y, z) = 0$. Thus $tR \subset N_l$ and $Rt \subset N_l$. Suppose that $t \in T$ and $w \in R$.

First note that $tw \cdot (x, y, z) = t \cdot w(x, y, z)$. But (3) multiplied on he left by t yields

$t \cdot w(x, y, z) = -t \cdot (w, x, y)z = -t(w, x, y) \cdot z = 0$.

Thus $tw \cdot (x, y, z) = 0$.

However (7) yields $(t, w)(x, y, z) = -(x, w)(t, y, z) = 0$.

This implies $tw \cdot (x, y, z) = wt \cdot (x, y, z)$.

Using $tw \cdot (x, y, z) = 0$, we obtain $wt \cdot (x, y, z) = 0$. At this point we have verified that T is an ideal of R. The rest is obvious. This completes the proof of the lemma. □

Lemma 2: If R is semiprime, then $T \cap I = 0$.

Proof. : Using Lemma 1 and the identity (3), we establish readily that $T.I = 0$. But $T \cap I$ is an ideal of R which squares to zero. Since R is semiprime, then $T \cap I = 0$. This completes the proof of the lemma. □

Lemma 3: If R is a semiprime ring, then
$((a, b, a), R) = 0$ (8)

$(y, z, (a, b)) = 0$ (9)

and $(y, z, (a, b, a)) = 0$. (10)

Proof :

Using (7) we see that

$((a, b, a), c)(x, y, z) = -(x, c)((a, b, a), y, z) = 0$, because of (1). This implies $((a, b, a), c) \subset T$. But this element is also clearly in I. Then lemma 2 implies $((a, b, a), c) = 0$.

i.e., $((a, b, a), R) = 0$.

Using the linearization of (1) and (2) we obtain

$((x,y, (a,b)), r,s) = -(((a,b),y,x), r,s) = 0$.

Thus $(x, y, (a, b))$ is an element of N_l. Also (3) implies

$z(x, y, (a, b)) = (zx, y, (a, b)) - (z, xy, (ab)) + (z, x, y(a, b)) - (z, x, y)(a, b)$.

Hence $(z(x,y,(a,b)), r,s) = ((z,x,y(a,b)), r,s) - ((z,x,y)(a,b), r,s)$. However $(a,b) \subset N_l$ because of (2), so that using (6) we get $-((z,x,y)(a,b), r,s) = -(a,b)((z,x,y), r,s)$.

Also $((z,x,y(a,b)), r,s) = -((y(a,b),x,z), r,s) = -((a,b)(y,x,z), r,s) = -(a,b)((y,x,z), r,s)$,

using (6). Thus $(z(x, y, (a, b)), r, s) = -(a, b)((y, x, z) + (z, x, y), r, s) = 0$, using a linearization of (1).

But then (6) implies $(x, y, (a, b))\ (z, r, s) = 0$, so that $(x, y, (a, b)) \subset T$. Since this element is also an associator, it obviously is also in I. Thus lemma 2 applies and we obtain $(y, z, (a, b)) = 0$. Using the linearization of (1) and (2) we obtain

$$((x,y,(a,b,a)),r,s) = -(((a,b,a),y,x),r,s) = 0.$$

Thus $(x, y, (a, b, a))$ is an element of N_l. Also (3) implies

$$z(x,y,(a,b,a)) = (zx,y,(a,b,a)) - (z,xy,(a,b,a)) + (z,x,y(a,b,a)) - (z,x,y)\ (a,b,a).$$

Hence $(z(x,y(a,b,a)),r,s) = ((z,x,y(a,b,a)),r,s) - ((z,x,y)\ (a,b,a),r,s)$.

However $(a, b, a) \subset N_l$, because of (1). So that using (6)

we get $-((z,x,y)\ (a,b,a),r,s) = -(a,b,a)\ ((z,x,y),r,s)$. Also $((z, x, y(a, b, a)), r, s) = -((y(a, b, a), x, z), r, s) = -((a, b, a)\ (y, x, z), r, s) = -(a, b, a)\ ((y, x, z), r, s)$ using (6).

Thus

$(z(x, y,(a, b, a)), r, s) = -(a, b, a)\ ((y, x, z) + (z, x, y),r, s) = 0$, using a linearization of (1). But then (6) implies

$(x, y,(a, b, a))\ (z, r, s) = 0$, so that $(x, y, (a, b, a)) \subset T$. Since this element is also an associator, it obviously is also in I. Thus lemma 2 applies and we obtain $(y, z, (a, b, a)) = 0$. This completes the proof of the lemma. □

Lemma 4 : If R is a semiprime ring, then R is flexible,

i.e., $(x, y, x) = 0$.

Proof : By taking $w = a$, $x = (a, b, a)$, $y = a$ and $z = b$ in (3), we get $(a(a,b,a),a,b) = (a,(a,b,a)a,b)$. (11)

We know that the following identity is valid in any ring.

$(xy, z) = x(y, z) + (x, z)y + (x, y, z) + (z, x, y) - (x, z, y)$. Using (7) and (8) in this equation, we obtain

$$(a,b,a(a,b,a)) + (a(a,b,a),a,b) - (a,a(a,b,a),b) = 0. \quad (12)$$

Substituting (11) in (12), we get

$(a, b, a(a, b, a)) = 0$. (13)

By taking $w = a$, $x = b$, $y = a$, $z = (a, b, a)$ in (3) and using (10) and (13), we get

$$(a, b, a)^2 = 0. \quad (14)$$

By using the same method as in lemma 3, from (1) and (10), it follows that $(x, (a, b, a), y) = 0$. So (a, b, a) is in the center of R, because of (8). We know that a center element squares to zero generates an ideal which squares to zero. Since R is semiprime, (14) implies that $(a, b, a) = 0$. i.e., R is flexible. This completes the proof of the lemma. □

Lemma 5 : If R is a semiprime ring, then $(x , (a, b), y) = 0$.

Proof : In any ring we have

$$(xy, z) - x(y, z) - (x, z)y =$$

$$(x, y, z) + (z, x, y) - (x, z, y).$$

By putting $z = x$ in this equation, we get

$$(xy, x) + x(x, y) = (x, y, x).$$

By forming the associators both sides we obtain

$((x, y, x), r, s) = (x (x, y), r, s) +((xy, x), r, s)$ or $((x, y, x), r, s) = ((x, y)x, r, s)$.This implies$((x y, x), r, s) = (x, y) (x, r, s) = 0$.

i.e., $(x, y) (x, r, s) = 0$. Linearizing this with $x = x+x'$,

we obtain $(x, y) (x', r, s) + (x', y) (x, r, s) = 0$.

If we substitute a commutator v for x', we see that

$(v, y) (x, r, s) = 0$, using (2).In any ring

$$((x, y), z) + ((y, z), x) + ((z, x), y) = S(x, y, z) - S(x, z, y).$$

So$(x,y),z)(q,r,s)+((y,z),x)(q,r,s)+((z,x),y)(q,r,s)=$

$S(x,y,z)(q,r,s)-S(x,z,y)(q,r,s)$. This implies $S(x,y,z) (q,r,s) - S(x,z,y) (q,r,s) = 0$.From this, we get $2((x,y,z) (q,r,s)+(y,z,x) (q,r,s) + (z,x,y) (q,r,s)) = 0$. If we put z as a commutator, then $(x, y, (a,b)) (q, r, s) + (y, (a, b), x) (q, r, s) + ((a, b), x, y) (q, r, s) = 0$. Using (9) and (2) $(y,(a, b),x) (q, r, s) = 0$.

So $(y,(a, b),x) \subset T$ and in I.

Thus $(x,(a, b),y) \in T \cap I$. Using lemma 2, we have $(x,(a, b),y) = 0$. This completes the proof of the lemma. □

Theorem 1: R must be a subdirect sum of a semiprime associative ring and a semiprime commutative ring.

Proof : It follows from lemma 3 and lemma 5 that R satisfies Thedy's hypothesis [6]. Hence the ring R is a subdirect sum of a semiprime associative ring and a semiprime commutative ring. This completes the proof of the theorem. □

3. REFERENCES

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