# EXPLORING COMPLEX 3X+1MAPPING

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*Abstract: The well known $3x+1$ function $f : Z \to Z$ defined as*

$$f(x) = \begin{cases} 3x+1 & \text{if x is odd} \\ \frac{x}{2} & \text{if x is even} \end{cases}$$

*is generalised to complex $3x+1$ Mapping $F : C \to C$ based on investigating the divisors of ceiling ($|z|$)where $z$ is a complex number.The equivalent conjecture to the complex function F is formulated and the main result namely "For each complex number z, applying successive iterations of F, eventually reaches 1", is discussed*

*Keywords : $3x+1$ conjecture,Collatz problem, complex $3x+1$ mapping.*

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## 1. INTRODUCTION

The $3x+1$ problem is a popular problem in Number theory concerning iteration of the function defined $f : Z \to Z$ as

$$f(x) = \begin{cases} 3x+1 & \text{if x is odd} \\ \frac{x}{2} & \text{if x is even} \end{cases} \tag{1}$$

The $3x+1$ conjecture asserts that, for each $x \in Z^+$, there is a $n \in Z^+$ such that $f^n(x) = 1$. This problem is also known as Collatz problem.

The trajectory of $x$ under $f$ or the orbit of $x$ is the sequence of iterates, $x, f(x), f^2(x), \ldots, f^n(x), \ldots$ where $f^n$ denotes $n$-fold composition of $f$ with itself.

The $3x+1$ function f in (1) is generalized to the complex function $F : C \to C$ defined as

$$F(z) = \begin{cases} 3z+1 & \text{if ceiling}(|z|)\text{isodd} \\ \frac{z}{2} & \text{ifceiling}(|z|)\text{iseven} \end{cases} \tag{2}$$

where $|z| = [Re(z)^2 + Im(z)^2]^{\frac{1}{2}}$ and ceiling($|z|$) represents the least integer greater than or equal to $|z|$ [3]. This function is called as complex Collatz function.

The equivalent conjecture corresponding to (2) is "The triconvergence property holds for all $n$",where the triconvergence property of z is " The trajectory of z under F can be partitioned into three subsequences a,b,c such that $a \to 1, b \to 4, c \to 2$ "[3].

It is also remarked that to obtain a generalisation one looks for sets of complex numbers satisfying the triconvergence property and one need not look for to find complex numbers that probably do not satisfy the triconvergence property.The initial value $3+5i$ have grown to about $1.25 \times 10^{12} + 1.42 \times 10^{12} i$, after $10^5$ iterations of F, is very unlikely that $3+5i$ satisfies the triconvergence property.[3]

In this paper, we generalise, the $3x+1$ function $f : Z \to Z$ in (1) to complex function $F : C \to C$ for whole complex plane, with no constraints, based on investigating the divisors of ceiling( $|z|$ ), which are integers, where $z$ is a complex number. The equivalent conjecture to the complex function F is formulated and the main result namely "For each complex number z, applying succesive iterations of F, eventually reaches 1", is discussed

***Complex $3x+1$ Function:***

We consider (2) here, that is the function $F : C \to C$ defined as

$$F(z) = \begin{cases} 3z+1 & \text{if ceiling}(|z|) \text{is odd} \\ \frac{z}{2} & \text{if ceiling}(|z|) \text{is even} \end{cases}$$

Also the orbit of z under F is the same as the orbit of x under $f$ [3].

Throughout our discussions, $z_0$ represents a complex number and $n$ represents a positive integer.

To start with, the set $C$ of complex numbers is partitioned into the following residue classes.[4]. $(a) := \{z_0 \in C : \text{ceiling}(|z_0|) \text{ is odd and is equal to } 1 \bmod 4\}$

$$(b) := \{z_0 \in C : \text{ceiling}(|z_0|) \text{is odd and is equal to} -1 \bmod 4\}$$

$$(c) := \{z_0 \in C : \text{ceiling}(|z_0|) \text{is even}\}$$

The residue classes (a),(b) and (c) are called respectively orbits A,B and C.

***Lemma 1:*** *If $z_0$ is in (a) and ceiling( $|z_0|$ ) $\geq 5$ then { ceiling( $|z_0|$ ) $-1$ } is divisible by at least 4 .*

**Proof:**

As $\text{ceiling}(|z_0|) \equiv 1 \bmod 4$ and $\frac{\text{ceiling}(|z_0|)+1}{2}$ is equal to 2k+1, for k = 1,2,3,..., we have $\text{ceiling}(|z_0|)-1 = 4k$ for $k = 1,2,3....$ which completes the proof.

In order to apply succesive iterations of $F$, we rewrite F(z) as
$F(z) = 3z+1 = 3(z-1)+4$ when ceiling $(|z|)$ is odd. (3)
Let $z_0 \in$ (a). Then by Lemma 1, we get

$$\frac{3(z_0-1)}{4}+1 = \frac{F(z_0)}{4}; F^3(z_0) \quad (4)$$

which we denote by $G(z_0)$.

$$\text{Hence } G(z_0) = \frac{3(z_0-1)}{4}+1 \text{ if } z_0 \in (a) \quad (5)$$

The orbit A at $z_0$ consists $z_0$, $G(z_0), G^2(z_0)$ .... such that $G^i(z_0)$ for all $i$ are in (a).

***Theorem:1*** *Let* $z_0 \in (a)$ *.Then ceiling* $|G(z_0)| \equiv$ *0 mod 3 or 1 mod 3 or 2 mod 3.*

**Proof:**

$$z_0 \in (a) \Rightarrow \text{ceiling}\,|z_0| = 4k+1, \;\; k > 0$$

$$\Rightarrow 4k < |z_0| \le 4k+1 \quad (6)$$

That is $z_0 \in (4k, \;\; 4k+1]$

$$|G(z_0)| = \left|\frac{3z_0+1}{4}\right| = \frac{|3z_0+1|}{4} \le \frac{3|z_0|+1}{4} \quad (7)$$

$$4k < |z_0| \le 4k+1 \Rightarrow 12k < 3|z_0| \le 12k+3$$

$$\Rightarrow \frac{12k+1}{4} < \frac{3|z_0|+1}{4} \le \frac{12k+4}{4}$$

$$\Rightarrow 3k + \frac{1}{4} < \frac{3|z_0|+1}{4} \le 3k+1.$$

$$\Rightarrow \frac{3|z_0|+1}{4} \in (3k+\frac{1}{4}, 3k+1]. \quad (8)$$

Hence ceiling{ $\frac{3|z_0|+1}{4}$ }= 3k+1.

By (7) $\text{ceiling}(|G(z_0)|) \le \text{ceiling}\left\{\frac{3|z_0|+1}{4}\right\} = 3k+1.$ (9)

Also we have $G(z_0) = \frac{3(z_0-1)}{4}+1$ by (5).

$$|G(z_0)| = \left|\frac{3(z_0-1)}{4}+1\right|$$

$$\le \frac{3|z_0-1|}{4}+1$$

Hence $\text{ceiling}(|G(z_0)|) \le \text{ceiling}\left[\frac{3|z_0-1|}{4}+1\right]$ (10)

$$||z_0|-1| \le |z_0-1| \Rightarrow \frac{3}{4}||z_0|-1|+1 \le \frac{3}{4}|z_0-1|+1$$

Hence $\text{ceiling}\{\frac{3}{4}||z_0|-1|+1\} \le \text{ceiling}\{\frac{3}{4}|z_0-1|+1\}$ (11)

$$4k < |z_0| \le 4k+1 \Rightarrow 4k-1 < |z_0|-1 \le 4k.$$

$$\Rightarrow \frac{3}{4}(4k-1) < \frac{3}{4}\{|z_0|-1\} \le \frac{3}{4}(4k)$$

$$\Rightarrow 3k - \frac{3}{4} < \frac{3}{4}\{|z_0|-1\} \le 3k$$

$$\Rightarrow 3k + \frac{1}{4} < \frac{3}{4}||z_0|-1|+1 \le 3k+1$$

$$z_0 \in (4k, 4k+1] \Rightarrow \{\frac{3}{4}||z_0|-1|+1\} \in (3k+\frac{1}{4}, 3k+1]$$

Thereforeceiling$\{\frac{3}{4}||z_0|-1|+1\} = 3k+1$ (12)

By (10) and (11), we have the following possibilities.

(i)ceiling $(|G(z_0)|)$ = ceiling $\left(\frac{3}{4}||z_0|-1|+1\right)$ = 3k+1

(ii)ceiling $(|G(z_0)|)$ < ceiling $\left(\frac{3}{4}||z_0|-1|+1\right)$ = 3k+1

(iii)ceiling $(|G(z_0)|)$ > ceiling $\left(\frac{3}{4}||z_0|-1|+1\right)$ = 3k+1 (13) (13)

By (9) and (13),we have ceiling $(|G(z_0)|) \leq 3k+1$ (or)
ceiling $(|G(z_0)|) > 3k+1$.
claim:1 ceiling $(|G(z_0)|)$ cannot be less than $3k$.
For: Suppose ceiling $(|G(z_0)|) < 3k$.

$$\Rightarrow |G(z_0)| \leq 3k-1$$
$$\Rightarrow \left|\frac{3z_0+1}{4}\right| \leq 3k-1$$
$$\Rightarrow |3z_0+1| \leq 12k-4 \qquad (14)$$

$$\text{Also } |3z_0+1| \leq 3|z_0|+1 \qquad (15)$$

Therefore by (14) and (15) we have the following possibilities.

(i) $3|z_0|+1 = 12k-4$, (ii) $3|z_0|+1 < 12k-4$, (iii) $3|z_0|+1 > 12k-4$.

If $3|z_0|+1 = 12k-4$, then $|z_0| = \frac{12k-5}{3} = (4k-1) - \frac{2}{3}$, which is a condradiction to $|z_0| \in (4k, 4k+1]$

(or) ceiling $(|z_0|) = 4k-1$, which is a contradiction to ceiling $(|z_0|) = 4k+1$.

If $3|z_0|+1 < 12k-4$, then $|z_0| < (4k-1) - \frac{2}{3}$, which is a contradiction to $|z_0| \in (4k, 4k+1]$ (or) ceiling $(|z_0|) \leq 4k-1$, a contradiction to ceiling $(|z_0|) = 4k+1$.

If $3|z_0|+1 > 12k-4, |z_0| > (4k-1) - \frac{2}{3}$.

Therefore ceiling $(|z_0|) \geq 4k-1$.

If ceiling $(|z_0|) = 4k-1$ then $z_0 \in (b)$, a contradiction to $z_0 \in (a)$
If ceiling $(|z_0|) = 4k$ then $z_0 \in (c)$, a contradiction to $z_0 \in (a)$
If ceiling $(|z_0|) = 4k+1$ then $3|z_0|+1 = 3(4k+1)+1$, a condraction to $3|z_0|+1 > 12k+4$.

If ceiling $(|z_0|) \geq 4k+2 \Rightarrow z_0 \notin$ (a). Hence claim 1.
claim:2 ceiling $(|G(z_0)|)$ cannot be greater than $3k+2$.
For: Suppose ceiling $(|G(z_0)|) > 3k+2$.

$$\Rightarrow |G(z_0)| > 3k+2$$

$\Rightarrow \left| \frac{3z_0 + 1}{4} \right| > 3k + 2$

$\Rightarrow | 3z_0 + 1 | > 12k + 8$

Hence $3 | z_0 | + 1 \geq | 3z_0 + 1 | > 12k + 8 \Rightarrow | z_0 | > \frac{12k+7}{3} = (4k+2) + \frac{1}{3}$, a contradiction to $| z_0 | \in (4k, 4k+1]$ (or) ceiling $(| z_0 |) > 4k+2$; a contradiction to ceiling $(| z_0 |) = 4k+1$.

Hence claim 2 and hence the Theorem 1.

**Example:1** $z_0 = 35 + 67.5i$
ceiling $(| z_0 |) = 77 \equiv 1 \bmod 4 \Rightarrow z_0 \in (a)$.
We have $G(z_0) = 26.5 + 50.625i$
ceiling $(| G(z_0) |) = 58 \equiv 1 \bmod 3$.

**Example:2** $z_0 = 4 + 16i$
ceiling $(| z_0 |) = 17 \equiv 1 \bmod 4 \Rightarrow z_0 \in (a)$.
Now $G(z_0) = 3.25 + 12i$
ceiling $(| G(z_0) |) = 13 \equiv 1 \bmod 3$.

**Example:3** $z_0 = -132 - 256i$
ceiling $(| z_0 |) = 289 \equiv 1 \bmod 4$.
Therefore $G(z_0) = -98.75 - 192i$
ceiling $(| G(z_0) |) = 216 \equiv 0 \bmod 3$.

**Lemma:2** *If $z_0$ is in (b) and ceiling $(| z_0 |) \geq 3$ then $\frac{\text{ceiling}(| z_0 |) - 1}{2}$ is odd.*

**Proof:**

As ceiling $(| z_0 |) \equiv -1 \bmod 4$ and $\frac{\text{ceiling}(| z_0 |) + 1}{4}$ is even equal to $2k$ for $k = 1,2,3....$ We have $\frac{\text{ceiling}(| z_0 |) - 1}{2}$ is odd, equal to $2k+1$ for $k = 1,2,3...$ which completes the proof.

Let $z_0$ be in (b).By (3) we have

$$F(z_0) = \frac{3(z_0 - 1)}{2} + 2 = \frac{F(z_0)}{2} = F^2(z_0) \tag{16}$$

which we denote by $H(z_0)$.

Hence $H(z_0) = \dfrac{3(z_0 - 1)}{2} + 2$ if $z_0$ is in (b) (17)

The orbit B at $z_0$ consists $z_0, H(z_0), H^2(z_0)\ldots$ such that $H^i(z_0)$ for all $i$ are in (b).

**Theorem:2** Let $z_0 \in$ (b). Then ceiling $(|H(z_0)|) \equiv$ 0 mod 3 or 1 mod 3 or 2 mod 3.

***Proof:***

$$z_0 \in (b) \Rightarrow \text{ceiling}(|z_0|) = 4k - 1, k > 0$$

$$\Rightarrow 4k - 2 < |z_0| \le 4k - 1 \quad (18)$$

That is $z_0 \in (4k - 2,\ 4k - 1]$

$$|H(z_0)| = \left|\frac{3z_0 + 1}{2}\right| \le \frac{3|z_0| + 1}{2} \quad (19)$$

$$4k - 2 < |z_0| \le 4k - 1 \Rightarrow 12k - 6 < 3|z_0| \le 12k - 3$$

$$\Rightarrow \frac{12k - 6 + 1}{2} < \frac{3|z_0| + 1}{2} \le \frac{12k - 3 + 1}{2}.$$

$$\Rightarrow 6k - 3 + \frac{1}{2} < \frac{3|z_0| + 1}{2} \le 6k - 1$$

$$\Rightarrow 3(2k - 1) + \frac{1}{2} < \frac{3|z_0| + 1}{2} \le 3(2k - 1) + 2.$$

$$\le 3(2k - 1) + 2. \quad (20)$$

That is $3m + \frac{1}{2} < \frac{3|z_0| + 1}{2} \le 3m+2$ where m = 2k-1. Therefore $\frac{3|z_0| + 1}{2} \in (3m + \frac{1}{2}, 3m + 2]; (6k - 3 + \frac{1}{2},\ 6k - 1]$

Hence ceiling $\left(\frac{3|z_0| + 1}{2}\right) = 3m + 2; 6k - 1$.

$$\text{By(19)ceiling}(|H(z_0)|) \le \text{ceiling } \left(\frac{3|z_0| + 1}{2}\right) = 3m + 2 \quad (21)$$

Also we have $H(z_0) = \dfrac{3(z_0 - 1)}{2}$ +2 by (17).

$$|H(z_0)| = \left|\frac{3(z_0 - 1)}{2} + 2\right|$$

$$\le \frac{3|z_0 - 1|}{2} + 2.$$

$$\Rightarrow \text{ceiling}\ \ (|H(z_0)|) \le \text{ceiling}[\frac{3|z_0-1|}{2}+2] \tag{22}$$

$$||z_0|-1| \le |z_0-1| \Rightarrow \frac{3}{2}||z_0|-1|+2 \le \frac{3}{2}|z_0-1|+2$$

$$\Rightarrow \text{ceiling}\ \ (\frac{3}{2}||z_0|-1|+2) \le \text{ceiling}(\frac{3}{2}|z_0-1|+2) \tag{23}$$

$$4k-2 < |z_0| \le 4k-1 \Rightarrow 4k-3 < |z_0|-1 \le 4k-2$$

$$\Rightarrow \frac{3}{2}(4k-3) < \frac{3}{2}||z_0|-1| \le \frac{3}{2}(4k-2)$$

$$\Rightarrow \frac{12k-9}{2} < \frac{3}{2}||z_0|-1| \le \frac{12k-6}{2}$$

$$\Rightarrow \frac{12k-5}{2} < \frac{3}{2}||z_0|-1|+2 \le \frac{12k-2}{2}$$

$$\Rightarrow 6k-2-\frac{1}{2} < \frac{3}{2}||z_0|-1|+2 \le 6k-1$$

That is $3(2k-1)+\frac{1}{2} < \frac{3}{2}||z_0|-1|+2 \le 3(2k-1)+2$

That is $3m+\frac{1}{2} < \frac{3}{2}||z_0|-1|+2 \le 3m+2$

Therefore $z_0 \in (4k-2, 4k-1] \Rightarrow \frac{3}{2}||z_0|-1|+2 \in (3m+\frac{1}{2}, 3m+2)$

$$\text{Therefore ceiling } [\frac{3}{2}||z_0|-1|+2] = 3m+2 \tag{24}$$

By (22) and (23) we have the following possibilities

(i)ceiling $(|H(z_0)|) = \text{ceiling}\left(\frac{3}{2}||z_0|-1|+2\right) = 3m+2$

(ii)ceiling $(|H(z_0)|) < \text{ceiling}\left(\frac{3}{2}||z_0|-1|+2\right) = 3m+2$

$$\text{(iii)ceiling } (|H(z_0)|) > \text{ceiling}\left(\frac{3}{2}||z_0|-1|+2\right) = 3m+2\ (25 \tag{25}$$

By (21) and (25) we have ceiling $(|H(z_0)|) \le$ 3m+2 (or) ceiling $(|H(z_0)|) > 3m+2)$

claim:1 ceiling$(|H(z_0)|$ cannot be less than 3m.
For: Suppose ceiling $(|H(z_0)|) < 3m$.

$$\Rightarrow |H(z_0)| \le 3m-1.$$

That is $\left|\frac{3z_0+1}{2}\right| \le 3m-1$.

Therefore $|3z_0+1| \le 6m-2$ (26)

Also $|3z_0+1| \le 3|z_0|+1$ (27)

By (26) and (27) we have $3|z_0|+1 = 6m-2$
(or) $3|z_0|+1 < 6m-2$
(or) $3|z_0|+1 > 6m-2$

If $3|z_0|+1 = 6m-2$ then $|z_0| = 2m-1 = 2(2k-1)-1 = 4k-3$, and hence ceiling $(|z_0|) = 4k-3$, which is a contradiction to ceiling$(|z_0|) = 4k-1$.

If $3|z_0|+1 < 6m-2$ then $|z_0| < 4k-3$, and hence ceiling $(|z_0|) \le 4k-3$, which is a contradiction to ceiling $(|z_0|) = 4k-1$.

If $3|z_0|+1 > 6m-2$ then $|z_0| > 4k-3$, and hence ceiling $(|z_0|) \ge 4k-2$.

If ceiling $(|z_0|) = 4k-2$ then $z_0 \in (c)$, which is a contradiction to $z_0 \in (b)$.

If ceiling $(|Z_0|) = 4k-1$ then $3|z_0|+1 = 3(4k-1)-1$
$= 12k-4$
$= 6(2k-1)+2$
$= 6m+2$
a contradiction to $3|z_0|+1 > 6m-2$.
If ceiling $(|z_0|) \ge 4k$ then $z_0 \notin (b)$. Hence claim:1
claim:2 ceiling$(|H(z_0)|)$ cannot be greater than $3m+2$.
For: Suppose ceiling $(|H(z_0)|) > 3m+2$.

$$\Rightarrow |H(z_0)| > 3m+2$$
$$\Rightarrow \left|\frac{3z_0+1}{2}\right| > 3m+2$$
$$\Rightarrow |3z_0+1| > 6m+4$$

Hence $3|z_0|+1 \ge |3z_0+1| > 6m+4$.

$\Rightarrow |z_0| > \frac{6m+3}{3} = 2m+1; 2(2k-1)+1 = 4k-1$, which is a contradiction to $|z_0| \in (4k-2,\ 4k-1]$ (or) ceiling $(|z_0|) > 4k-1$, a contradiction to ceiling $(|z_0|) = 4k-1$.

Hence claim 2 and hence the Theorem 2.

**Example:5** $z_0 = 13889 + 7638i$
ceiling ($|z_0|$) = 15851 ≡ -1 mod 4
$\Rightarrow z_0 \in (b)$
We have $H(z_0) = 20834 + 11457i$
ceiling($|H(z_0)|$) = 23777 ≡ 2 mod 3

**Example:6** $z_0 = 15 + 506i$
ceiling($|z_0|$) = 507 ≡ -1 mod 4 (4(127)-1).
$H(z_0) = 23 + 759i$
ceiling ($|H(z_0)|$) = 760 ≡ 1 mod 3.

**Theorem:3** Let $z_0 \in (b)$. If ceiling ($|z_0|$) + 1 is divisible by $2^n$ not by $2^{n+1}$ and each $H^i(z_0), 1 \le i \le n-2$ is in (b) then $H^{n-1}(z_0)$ is in (a).

**Proof:**

$z_0$ is in (b) $\Rightarrow \dfrac{\text{ceiling}(|z_0|)+1}{2}$ is even.

We have $H(z_0) = \dfrac{3(z_0 - 1)}{2} + 2$ by (17)

If ceiling ($|H(z_0)|$) is in (b) we have $H^2(z_0) = \dfrac{3^2(z_0 - 1)}{2^2} + \dfrac{3}{2} + 2$, proceeding, after n-1 iterations, as ceiling ($|H^{n-2}(z_0)|$) is in (b), We obtain

$$H^{n-1}(z_0) = \frac{3^{n-1}}{2^{n-1}}(z_0 - 1) + (\frac{3}{2})^{n-1} + \ldots + (\frac{3}{2}) + 2.$$

$$\Rightarrow H^{n-1}(z_0) + 1 = \frac{3^{n-1}}{2^{n-1}}(z_0 - 1) + \frac{3^{n-1}}{2^{n-2}}.$$

$$= \frac{3^{n-1}}{2^{n-1}}(z_0 + 1). \tag{28}$$

claim: $\dfrac{\text{Ceiling}(|H^{n-1}(z_0)|)+1}{2}$ is odd.

For $|H^{n-1}(z_0) + 1| \le |H^{n-1}(z_0)| + 1.$ (29)

and $\left|\dfrac{3^{n-1}(z_0+1)}{2^{n-1}}\right| \le \dfrac{3^{n-1}}{2^{n-1}}(|z_0|+1).$ (30)

By (28) therefore, $|H^{n-1}(z_0)+1| = \left|\dfrac{3^{n-1}}{2^{n-1}}(z_0+1)\right|$

$$\le \frac{3^{n-1}}{2^{n-1}}(|z_0|+1). \quad (31)$$

By (29) and (31) we have the following possibilities

(i) $|H^{n-1}(z_0)|+1 = \dfrac{3^{n-1}}{2^{n-1}}(|z_0|+1)$

(ii) $|H^{n-1}(z_0)|+1 < \dfrac{3^{n-1}}{2^{n-1}}(|z_0|+1)$

(iii) $|H^{n-1}(z_0)|+1 > \dfrac{3^{n-1}}{2^{n-1}}(|z_0|+1)$ (32)

$$\text{(i))} \Rightarrow \text{ceiling}\left[|H^{n-1}(z_0)|+1\right] = \text{ceiling}\left[\frac{3^{n-1}}{2^{n-1}}(|z_0|+1)\right]$$

That is ceiling $(|H^{n-1}(z_0)|+1 \le 3^{n-1}\left[\dfrac{\text{ceiling}(|z_0|)+1}{2^{n-1}}\right]$

Therefore $\dfrac{\text{ceiling }(|H^{n-1}(z_0)|)+1}{2} \le 3^{n-1}\left[\dfrac{\text{ceiling}(|z_0|)+1}{2^n}\right]$

= odd(byhypothesis)

Similarly(32(ii)) $\Rightarrow \dfrac{\text{ceiling }(|H^{n-1}(z_0)|)+1}{2} < 3^{n-1}\left[\dfrac{\text{ceiling}(|z_0|)+1}{2^n}\right]$

= odd

(32(iii)) $\Rightarrow \text{ceiling}(|H^{n-1}(z_0)|)+12 > \text{ceiling}\left[\dfrac{3^{n-1}}{2^{n-1}}(|z_0|+1)\right]$ (33)

Since ceiling $\left[\dfrac{3^{n-1}}{2^{n-1}}|z_0|+1\right] \le 3^{n-1}\left[\dfrac{\text{ceiling}(|z_0|)+1}{2^{n-1}}\right]$,

we have again the following

$$\frac{\text{ceiling}(|H^{n-1}(z_0)|)+1}{2} = \text{or} < \text{or} > 3^{n-1}\left[\frac{\text{ceiling}(|z_0|)+1}{2^n}\right] = \text{odd}$$

Hence $\dfrac{\text{ceiling}(|H^{n-1}(z_0)|)+1}{2} \le$ odd. and $\dfrac{\text{ceiling}(|H^{n-1}(z_0)|)+1}{2} \ge$ odd.

Hence $\frac{\text{ceiling}(|H^{n-1}(z_0)|)+1}{2}$ = odd, which completes the claim.

Therefore $H^{n-1}(z_0)$ is in (a).

**Example:** Let $z_0 = 238.75 + 3.0i$

ceiling($|z_0|$) = ceiling(| 238.7688474... |) = 239 ≡ -1 mod 4 ⇒ $z_0 \in$ (b)

Also ceiling $(|z_0|)+1$ is divisible by $2^4$, not by $2^5$.

Therefore $H^3(z_0) = (H^{n-1}(z_0)) = 808.15675 + 10.125$

where ceiling $(|H^3(z_0)|) = 809 \equiv 1 \bmod 4$

$\Rightarrow H^3(z_0) \in (a)$.

**Remark:**

In the above Theorem 3, the condition each $H^i(z_0)$, $1 \le i \le n-2$ is necessary. Otherwise $H^j(z_0) \in (c)$ for some $j, 1 \le j \le n-2$.

For example Let $z_0 = 72 + 104i$. ceiling $(|z_0|) = 127 \equiv -1 \bmod 4 \Rightarrow z_0 \in (b)$

Also ceiling $(|H(z_0)|)+1$ is divisible by $2^7$ not by $2^8$.

We have $H(z_0) = 108.5 + 156i$

Now ceiling $(|H(z_0)|) = 191 \equiv -1 \bmod 4 \Rightarrow H(z_0) \in (b)$

Therefore we have $H^2(z_0) = 163.5 + 234i$.

Now ceiling $(|H^2(z_0)|) = 286 \equiv 1 \bmod 3$, even.

Therefore $H^2(z_0) \in (c)$.

***Main Result:***

Here we discuss the main result which is equivalent to $3x+1$ conjecture corresponding to the mapping (2).

**Orbits:** The residue classes (a) (b) and (c) are called orbits A, B, and C respectively. The orbit A of $z_0$ under G consists $z_0$, $G(z_0)$, $G^2(z_0)$,... such that $G^i(z_0) \forall i$ are in $(a)$. Similarly the orbit B of $z_0$ under H consists $z_0$, $H(z_0)$, $H^2(z_0)$ such that $H^i(z_0) \forall i$ are in (b). The orbit C of $z_0$ under F consists $z_0$, $F(z_0)$, $F^2(z_0)$, ... such that $F^i(z_0) \forall$ i are in (c).

We remark that if $z_0$ is such that ceiling$(|z_0|) \equiv 0 \bmod 3$ or $1 \bmod 3$ or $2 \bmod 3$ then $z_0 \in$ any one of the residue classes (a),(b),(c).

**Theorem:4** Let $z_0$ be in (a). Then the successive iterations of $G(z_0)$ where $G(z_0)$ is given by (5), lead to any of the following cases.

$z_0$ in (a) implies ceiling ($|z_0|$) is odd and is equal to 1 mod 4.

case:1 If $G(z_0)$, $G^2(z_0)$, …,$G^{i-1}(z_0)$ are in (a) then $G^i(z_0) = \frac{3^i(z_0-1)}{4^i}+1$ (34)

By Theorem 1, we have ceiling ($|G^i(z_0)|) \equiv 0 \bmod 3 \text{ or } 1 \bmod 3 \text{ or } 2 \bmod 3$. That is ceiling($|G^i(z_0)|$) is any of the forms $3k+1, 3k+2, 3k$. Evidently ceiling ($|G^i(z_0)|$) may be odd or even integers as $z_0$ is in (a).

case:2 $G^i(z_0)$ is in (b). Then the iteration of H lead to

$$H^{n-1}(G^i(z_0)) = \frac{3^{n-1}\left(\frac{3^i(z_0-1)}{4^i}+2\right)}{2^{n-1}}-1 \qquad (35)$$

provided ceiling ($|G^i(z_0)|)+1$ is divisible by $2^n$, not by $2^{n+1}$ and each $H^j(G^i(z_0)), 1 \le j \le n-2$ is in (b). By Theorem 3 we have $H^{n-1}(G^i(z_0))$ is in (a).

case:3 $G^i(z_0)$ is in (c) and ceiling ($|G^i(z_0)|) = 3k+1$ for an odd integer $k \equiv -1 \bmod 4$. Then by Lemma 2, $k-1$ is divisible by 2. Hence the iteration of F gives $F(G^i(z_0)) = H\left(\frac{G^i(z_0)-1}{3}\right)$ (36)

$$\text{since } F(G^i(z_0)) = \frac{\left(\frac{3^i(z_0-1)}{4^i}+1\right)}{2} = 3\left[\frac{3^i(z_0-1)}{2\cdot 3\cdot 4^i}-\frac{1}{2}\right]+2$$

case:4 $G^i(z_0)$ is in (c) and ceiling ($|G^i(z_0)|) = 3k+1$ for an odd integer $k \equiv 1$ mod 4. Then by Lemma 1, $k-1$ is divisible by 4. Hence the iteration of F (twice) yield

$$F^2(G^i(z_0)) = G\left(\frac{G^i(z_0)-1}{3}\right) \qquad (37)$$

case:5 $G^i(z_0)$ is in (c) and ceiling ($|G^i(z_0)|$) is not equal to 1 mod 3. Note that ceiling ($|G^i(z_0)|$) is equal to either $3k$ or $3k+2$ for an even integer $k$. Then we have $F^n(G^i(z_0)) = \frac{G^i(z_0)}{2^n}$ provided ceiling ($|\frac{G^i(z_0)}{2^{n-1}}|$) is in (c) and not equal to 1 mod 3.

We remark that in case 3, case 4 and case 5, by iteration of F we mean the iteration of the function $F(z_0) = \frac{z_0}{2}$ when ceiling ($|z_0|$) is even as in (2).

**Theorem:5** Let $z_0$ be in (b) and $H(z_0)$ is given by (17). The successive iterations of $H(z_0)$ join with the orbit A.

**Proof:**

$z_0$ is in (b) $\Rightarrow$ ceiling($|z_0|$) is odd and is equal to -1 mod 4 say $4k-1$. If $H(z_0)$, $H^2(z_0)$ ...$H^{i-1}(z_0)$ are also in (b) then $H^i(z_0) = \frac{3^i(z_0+1)}{2^i} - 1$ by (28).

By Theorem 2, we have ceiling ($|H^i(z_0)|$) $\equiv 0 \bmod 3$ or $1 \bmod 3$ or $2 \bmod 3$. That is ceiling($|H(z_0)|$) is any of the forms $3m+1, 3m+2, 3m$ where $m = 2k-1 \forall k$. Evidently ceiling ($|H^i(z_0)|$) may be odd or even integers as $z_0$ is in (b).

case:1 As ceiling ($|z_0|$) + 1 is divisible by $2^i$ not by $2^{i+1}$ and each $H^j(z_0)$ $1 \le j \le i-1$ are in (b) then by Theorem 3 $H^i(z_0)$ is in (a). Further iteration of G lead to $G^{n-1}(H^i(z_0)) = \frac{3^{n-1}}{4^n-1}\{\frac{3^i(z_0+1)}{2^i} - 2\} + 1$

case:2 $H^i(z_0)$ is in (c) and ceiling ($|H^i(z_0)|$) is equal to $3m+1$ for an odd integer $m \equiv -1 \bmod 4$, by case 3 of Theorem 4 we get $F(H^i(z_0)) = H(\frac{H^i(z_0)-1}{3})$

case:3 $H^i(z_0)$ is in (c) and ceiling ($|H^i(z_0)|$) is equal to $3m+1$ for an odd integer $m \equiv 1 \bmod 4$ then by case 4 of Theorem 4 we get $F^2(H^i(z_0)) = G(\frac{H^i(z_0)-1}{3})$.

case:4 $H^i(z_0)$ is in (c) and ceiling ($|H^i(z_0)|$) is not equal to 1 mod 3. That is either 3m or 3m+2 for an even integer m, by case 5 of Theorem 4 we get $F^n(H^i(z_0)) = \frac{H^i(z_0)}{2^n}$.

We remark that case 1, case 2, case 3, and case 4 of Theorem 5 are nothing but case 1, case 3, case 4 and case 5 of Theorem 4, which completes the proof.

**Theorem:6** If $z_0$ is in (c) then the successive iterations of $F(z_0)$ where $F(z_0)$ is given by (2), join with the orbit A.

**Proof:**

$z_0$ is in (c) $\Rightarrow$ ceiling $(|z_0|)$ is even, either equal to 1 mod 3 or not equal to 1 mod 3.

case:1 If ceiling $(|z_0|) = 3k+1$ for an odd integer $k \equiv 1$ mod 4 then by case 4 of Theorem 4 we have $F^2(z_0) = G(\frac{z_0 - 1}{3})$.

Further if ceiling $(|F^i(z_0)|) \equiv 1$ mod 4 then the iterations of G lead to $G^n(F^i(z_0)) = \frac{3^n}{4^n}\{\frac{z_0 - 1}{3^i} - (\frac{3^i - 1}{2 \cdot 3^{i-1}})\} + 1$ for all $n$.

$$\text{Inparticular } G(F^i(z_0)) = \frac{\{\frac{z_0 - 1}{3^{i-1}} - (\frac{3^{i-2} - 1}{2 \cdot 3^{i-2}})\}}{4}$$

$$= F^2(F^{i-1}(z_0))$$

case:2 If ceiling $(|z_0|) = 3k+1$ for an odd integer $k \equiv -1$ mod 4 then by case 3 of Theorem 4 we have $F(z_0) = H(\frac{z_0 - 1}{3})$

Further if ceiling $(F^i(z_0)) \equiv -1$ mod 4 then the iterations of H lead to $H^n(F^i(z_0)) = \frac{3^n}{2^n}\{\frac{z_0 - 1}{3^i} - (\frac{3^i - 1}{2 \cdot 3^{i-1}})\} - 1$ provided ceiling $(|F^i(z_0)|)+1$ is divisible by $2^n$ not by $2^{n+1}$ and each $H^j(F^i(z_0))$ $1 \le j \le n-1$ is in (b). In particular $H(F^i(z_0)) = F(F^{i-1}(z_0))$. Hence by Theorem 3 we have $H^n(F^i(z_0))$ is in (a).

case:3 If ceiling $(|z_0|)$ is not equal to $3k+1$ for an odd integer k, that is either $3k$ or $3k+2$ for an even integer $k$, then we have $F^i(z_0) = \frac{z_0}{2^i}$ for all $i$, provided ceiling$(|\frac{z_0}{2^i}|)$ is not equal to $3k+1$ for an odd integer $k$.

Further if ceiling($|F^i(z_0)| \equiv 1 \bmod 4$ then $F^i(z_0)$ is in (a) and by case 1 of Theorem 4 we have $G^n(F^i(z_0)) = \frac{3^n(z_0 - 2^i)}{2^{2n+i}} + 1$. In particular $G(F^i(z_0)) = \frac{3z_0 + 2}{8}$. If ceiling $(|F^i(z_0)|) \equiv -1 \bmod 4$ then $F^i(z_0)$ is in (b) and by case 2 of theorem 4 we have $H^n(F^i(z_0)) = \frac{3^n(z_0 + 2^i)}{2^{n+i}} - 1$.

In particular $H(F(z_0)) = \frac{3z_0 + 2}{4}$. If ceiling$(|F^i(z_0)|)$ is even and is equal to $3k+1$ for an odd integer $k \equiv 1 \bmod 4$, we have $F^{i+2}(z_0) = G(\frac{F^i(z_0) - 1}{3})$ as well as for $k \equiv -1 \bmod 4$, we have $F^{i+1}(z_0) = H(\frac{F^i(z_0) - 1}{3})$ which is nothing but case 4 and case 3 of Theorem 4 respectively, which completes the proof.

***Properties of the functions G(z), H(z) and F(z).***

1. If $z_0$ is in (a) then $G(z_0) < z_0$ as the derivative of $G(z_0)$ with respect to $z_0$ is less than 1.
2. If $z_0$ is in (b) then $H(z_0) > z_0$ as the derivative of $H(z_0)$ with respect to $z_0$ is greater than 1.
3. If $z_0$ is in (c) then $F(z_0) < z_0$ as the derivative of $F(z_0)$ with respect to $z_0$ is less than 1.

**Theorem:7** If $z_0$ be in (a) such that ceiling $(|z_0|) = 1$ then the successive iterations of $z_0$ under G stays in the orbit A itself.

The theorem follows from property 1.

***Conjecture:***

For each $z_0$, applying successive iterations of F eventually reaches 1.(For practical purpose one can consider 1+0i ;$x+iy$ in C as 0.9 $\leq x \leq 1$ and y is in the neighbourhood of zero).

***Numerical Illustration*** The data was generated using iframe in netbeans 5.5 for java development environment and stored as text file.

1. Let $z_0 = 3 + 5i$, ceiling $(|z_0|) = 6 \equiv 0 \bmod 3. \Rightarrow z_0 \in$ (c). By Theorem 6, we have $z_1 = F(z_0) = 1.5 + 2.5i$ and ceiling$(|z_1|) = 3 \equiv 0 \bmod 3$ also -1 mod 4.

$\Rightarrow z_1 \in (b)$. By Theorem 5, we have $z_2 = H(z_1) = 2.75 + 3.75i$, ceiling $(|z_2|) = 5 \equiv 2 \bmod 3$,also $1 \bmod 4 \Rightarrow z_2 \in (a)$. By theorem 4, we have $z_3 = G(z_2) = 2.3125 + 2.8175i$, ceiling $(|z_3|) = 4 \equiv 1 \bmod 3$, even. $\Rightarrow z_3 \in (c)$. By Theorem 6, we have $z_4 = \frac{z_3 - 1}{3} = 0.4375 + 0.9375i$. As $\frac{\text{ceiling}|z_3| - 1}{3} = 1 \in (a)$ we have $z_5 = G(\frac{z_3 - 1}{3}) = 0.578125 + 0.703125i$ ceiling $(|z_5|) = 1 \Rightarrow z_5 \in (a)$. Now by Theorem 7, we have $z_{50} = 0.9999993 + 1.2584924E - 6i$ with ceiling $(|z_{50}|) = 1$.

By repeated iteration, we get $z_{99} = 0.9999999 + 9.502818E - 13 \approx 1 + 0i$.

2. Let $z_0 = 0 + 240.5i$, ceiling$(|z_0|) = 241 \equiv 1 \bmod 3$ also $1 \bmod 4 \Rightarrow z_0 \in$ (a). By Theorem 4, we have $z_1 = G(z_0) = 0.25 + 180.375i$, ceiling $(|z_1|) = 181 \equiv 1 \bmod 3$ also $1 \bmod 4$. $\Rightarrow z_1 \in (a)$ By Theorem 4, we have $z_2 = G(z_1) = 0.4375 + 135.28125i$ ceiling $(|z_2|) = 136 \equiv 1 \bmod 3$, even. $\Rightarrow z_2 \in (c)$.

Note that by case 1 of Theorem 4,we have $G^2(z_0) = \frac{3^2}{4^2}(z_0 - 1) + 1$

$= 0.4375 + 135.28125i$.

Now $z_3 = \frac{z_2 - 1}{3} = -0.1875 + 45.09375i$ and

$\frac{\text{ceiling}|z_2| - 1}{3} = 45 \equiv 1 \bmod 4$, by Theorem 6. We have

$z_4 = G(\frac{z_2 - 1}{3}) = 0.109375 + 33.820312i$ and ceiling$(|z_4|) = 34 \equiv 1 \bmod 3$ even. $\Rightarrow z_4 \in$ (c).By repeated process, we get $z_{11} = 0.42559814 + 3.566986i$, ceiling$(|z_{11}|) = 4 \Rightarrow z_{11} \in$ (c).

By Theorem 6, we get $z_{13} = 0.1063995336 + 0.8917465i$ ceiling $(|z_{13}|) = 1 \Rightarrow z_{13} \in$ (a).Now by theorem 7, we get

$z_{49} = 0.999991 + 8.967925E - 6i \approx 1+0i$

3. Let $z_0 = -318 - 4i.\text{ceiling}(|z_0|) = 319 \equiv$ 1 mod 3 also -1 mod 4. $\Rightarrow z_0 \in$ (b)

By Theorem 5, we have $z_1 = H(z_0) = -476.5 - 6i$, ceiling $(|z_1|) = 477 \equiv$ 1 mod 4, also 0 mod 3. $\Rightarrow z_1 \in$ (a). By Theorem 4 we have $z_2 = G(z_1) = -357.125 - 4.5i, \text{ceiling}(|z_2|) = 358 \equiv$ 1 mod 3 $\Rightarrow z_2 \in$ (c). By repeated process, we get $z_{18} = -3.384798 + 0.0500564458i, \text{ceiling}(|z_{18}|) = 4 \Rightarrow z_{18} \in$ (c)

$z_{19} = \frac{z_{18} - 1}{3} = -1.46159933 + 0.016685486i$ and $\frac{\text{ceiling}(|z_{19}|) - 1}{3} = 1 \in$ (a). we have by Theorem 1,

$z_{20} = -0.8461995 - 0.012514114i, \text{ceiling}(|z_{20}|) = 1$.

$\text{Now by Theorem 7, we have } z_{99} = 0.9999999 - 3.0033604E - 13i$

$\approx 1 + 0i.$

## 2. CONCLUSION

The complex function is defined by considering the other two variants, instead of ceiling $(|z|)$, namely floor $(|z|)$ = greatest integer $\leq |z|$ and round $(|z|)$ = integer nearest to $|z|$, rounding up incase of ambiguity and one can verify the 3 x +1 conjecture for the complex function with respect to floor $(|z|)$ and round $(|z|)$

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