AN ALTERNATE PROOF OF THE DE MOIVRE'S THEOREM.

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Abstract : Transformation is a method for solving the system, where we face difficult. As the well known transformations like Fourier transformation, Laplace transformation, Z-trans formations helps us to solve different equations like differential algebraic problems. In this paper I propose a new transformation called C-transformation, from the set of complex numbers onto the set of all 2×2 matrices of the form $\begin{pmatrix} a & ib \\ ib & a \end{pmatrix}$ where a, b are real numbers. I defined a new set M, which is nothing but the set of all 2×2 matrices of the form $\begin{pmatrix} a & ib \\ ib & a \end{pmatrix}$ where a, b are real numbers. The C-transformation is a transformation from the set of complex numbers onto M. I further show that the C-transformation is isomorphism from C onto M. The existence and uniqueness of such transformation is proved. By using this transformation, I try to give the geometrical interpretation for the matrix of the form $\begin{pmatrix} a & ib \\ ib & a \end{pmatrix}$ which is nothing but a point in the argand plane. I gave an alternate proof for the well known Demoiver's theorem. I further try to solve a conformal mapping from z-plane to w-plane using this C-transformation provided the angle preserving , ie $\theta = \phi$ (the angle between the curves in z-plane is equal to the angle between the curves in w-plane) with an example w=f(z)= z^2 at the point z=1+i.

Introduction : It is known that the set of real number system "R" is a ring with respect to general addition, general multiplication. ie (R,+,.) is a ring .In this ring (R,+,.) "1" is the identity element and "-1" is the additive inverse of "1". Now define a new set M, which is nothing but, the set of all 2×2 matrices of the form $\begin{pmatrix} a & ib \\ ib & a \end{pmatrix}$ where a, b are real numbers. Hence the set M is a ring with respect to matrix addition, matrix multiplication. In this ring "I" (unit matrix) is I=$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ the identity element and "-I", $-I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ is the additive inverse of "I", I=$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (unit matrix).

When we define a complex number Z, we define Z= (a+ib), where a, b are real numbers. And $i^2 = -1$, (additive inverse of 1). I suppose that there may be unique matrix J in the set M such that J^2 When we suppose J^2=-I, we should get J=$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$,

since $J^2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$=-I, $-I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ additive inverse of I, I=$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

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since $J^2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$=-I

{Of course, we may get 4 matrix in the form , satisfying the condition J^2=-I

1) When J=$\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$, $J^2 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$=-I

2) When J=$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $J^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$=-I

3) When J =$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$; $J^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$=-I

4) When J=$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, $J^2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$=-I }

among the four the forth matrix is only the matrix in M

Every complex number z=a+ib can be represented as a matrix m= aI+bJ ,a,b are real numbers I identity matrix and J=$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$

Proposition: There is a isomorphism from The set of complex number C into the set M of all 2×2 matrices of the form $\begin{pmatrix} a & ib \\ ib & a \end{pmatrix}$ where a , b are real numbers.

Proof: Suppose that f is a transformation from The set of complex number C into the set M of all 2×2 matrices of the form $\begin{pmatrix} a & ib \\ ib & a \end{pmatrix}$ where a , b are real numbers and Defined as f(Z)= $\begin{pmatrix} \frac{z+\bar{z}}{2} & \frac{z-\bar{z}}{2} \\ \frac{z-\bar{z}}{2} & \frac{z+\bar{z}}{2} \end{pmatrix}$ for z € C, When z=a+ib , $\frac{z+\bar{z}}{2}$ =a; $\frac{z-\bar{z}}{2}$ =bi then f(Z)= f (a+ib)= $\begin{pmatrix} a & ib \\ ib & a \end{pmatrix}$

As per the definition of the set M, $M=\{M_r / M_r$ is the matrix in the form $\begin{pmatrix} a & ib \\ ib & a \end{pmatrix}$ }

Proof Part I :-

Result :- We know that the set of all complex number system C is a field with respect to general addition and multiplication.

Claim 1) The set M is a ring with respect to matrix addition and matrix multiplication

For proving this, we already Know that M is a abelian group with respect to matrix addition and M is a group with respect to matrix addition i.e. (M,+) is an abelian group (M, .) is group And since

$$\begin{pmatrix} a_1 & ib_1 \\ ib_1 & a_1 \end{pmatrix}\begin{pmatrix} a_2 & ib_2 \\ ib_2 & a_2 \end{pmatrix}=\begin{pmatrix} a_2 & ib_2 \\ ib_2 & a_2 \end{pmatrix}\begin{pmatrix} a_1 & ib_1 \\ ib_1 & a_1 \end{pmatrix}$$

(verified)

Hence (M,.) is an abelian group.

Therefore 1) (M,+) is an abelian group

2)(M,.) is a semi group

3)Distributive properties holds good

Therefore (M,+,.) is ring under matrix addition and matrix multiplication.

Claim 2) The set M is a commutative ring with unit element ,here unit element is the unit matrix I.

Claim3) The ring (M,+,.) is a skew field.

Claim4) The ring (M,+,.) is a field since , every non zero element is invertable under matrix multiplication .

Claim 5) The ring (M,+,.) is withoutzero divisors, Hence (M,+,.) is Integral domain.

Proof Part II :

Claim :- The defined function f: $C\rightarrow M$ such that

f(Z)= $\begin{pmatrix} \frac{z+\bar{z}}{2} & \frac{z-\bar{z}}{2} \\ \frac{z-\bar{z}}{2} & \frac{z+\bar{z}}{2} \end{pmatrix}$ for z € C is an isomorphism

Proof 1) f is one- one and onto and well defined

Proof2) $f(z_1+z_2)=f(z_1)+f(z_2)$

Proof 3) $f(z_1z_2)=f(z_1)f(z_2)$

Note: when $z_1=z_2$, hence $f(z_1z_2)=f(z_1)f(z_1)=(f(z_1))^2$

Similarly,

$$f(z_1z_2z_3z_4.........z_n)=f(z_1)f(z_2)..........f(z_n)$$

If $z_1 = z_2 =............= z_n$

Hence

$$f(z_1z_1z_1z_1.........z_1)=f(z_1)f(z_1)..........f(z_1)=(f(z_1))^n$$

Definition: “C-Transformation”

If Z=a+ib then the c –transformation for Z=a+ib is

$$\begin{pmatrix} \frac{z+\bar{z}}{2} & \frac{z-\bar{z}}{2} \\ \frac{z-\bar{z}}{2} & \frac{z+\bar{z}}{2} \end{pmatrix}=\begin{pmatrix} a & ib \\ ib & a \end{pmatrix}$$

That is C-T(z)= C-T(a+ib)= $\begin{pmatrix} a & ib \\ ib & a \end{pmatrix}$

And the inverse transformation for $\begin{pmatrix} \frac{z+\bar{z}}{2} & \frac{z-\bar{z}}{2} \\ \frac{z-\bar{z}}{2} & \frac{z+\bar{z}}{2} \end{pmatrix}$ $=\begin{pmatrix} a & ib \\ ib & a \end{pmatrix}$ is a+ib =z i.e

$$C-T^{-1}(\begin{pmatrix} \frac{z+\bar{z}}{2} & \frac{z-\bar{z}}{2} \\ \frac{z-\bar{z}}{2} & \frac{z+\bar{z}}{2} \end{pmatrix})=C-T^{-1}(\begin{pmatrix} a & ib \\ ib & a \end{pmatrix})=$$

a+ib= z

Transformation on Addition of complex Numbers :

Let $z_1 = a_1 + i b_1$ and $z_2 = a_2 + i b_2$ and $z_1 + z_2$ is also a complex number then

C-transformation on $z_1 + z_2$ is

C-T($z_1 + z_2$)=C-T {(a_1+ib_1)+(a_2+ib_2)}

$$=\text{C-T}\{(a_1+a_2)+i(b_1+b_2)\}=\begin{pmatrix} a_1+a_2 & (b_1+b_2)i \\ (b_1+b_2)i & a_1+a_2 \end{pmatrix}$$

$$=\begin{pmatrix} a_1 & b_1 i \\ b_1 i & a_1 \end{pmatrix}+\begin{pmatrix} a_2 & b_2 i \\ b_2 i & a_2 \end{pmatrix}$$

And the Inverse C-transformation

$$C-T^{-1}\left(\begin{pmatrix} a_1 & b_1 i \\ b_1 i & a_1 \end{pmatrix}+\begin{pmatrix} a_2 & b_2 i \\ b_2 i & a_2 \end{pmatrix}\right)=C-T^{-1}\left(\begin{pmatrix} a_1 & b_1 i \\ b_1 i & a_1 \end{pmatrix}\right)+C-T^{-1}\left(\begin{pmatrix} a_2 & b_2 i \\ b_2 i & a_2 \end{pmatrix}\right)$$

$= (a_1 + i b_1) + (a_2 + i b_2)$

$= z_1 + z_2$

Let $z_1 = a_1 + i b_1$ and $z_2 = a_2 + i b_2$ then the C-Transformation on

$z_1 z_2 = (a_1 + i b_1)(a_2 + i b_2)$ is

C-T ($z_1 z_2$)=C-T(z_1).C-T(z_2)

$$=\begin{pmatrix} a_1 & b_1 i \\ b_1 i & a_1 \end{pmatrix}\begin{pmatrix} a_2 & b_2 i \\ b_2 i & a_2 \end{pmatrix}$$

$$=\begin{pmatrix} a_1a_2-b_1b_2 & i(a_1b_2+b_1a_2) \\ i(a_1b_2+b_1a_2) & a_1a_2-b_1b_2 \end{pmatrix}$$

And the inverse C-Tranformation for

$$\begin{pmatrix} a_1a_2-b_1b_2 & i(a_1b_2+b_1a_2) \\ i(a_1b_2+b_1a_2) & a_1a_2-b_1b_2 \end{pmatrix}=C-T^{-1}\left(\begin{pmatrix} a_1a_2-b_1b_2 & i(a_1b_2+b_1a_2) \\ i(a_1b_2+b_1a_2) & a_1a_2-b_1b_2 \end{pmatrix}\right)$$

$= (a_1a_2 - b_1b_2) + i(a_1b_2 + b_1a_2)$.

By the above definition of addition and multiplication of complex numbers, we can easily verify all others axioms.

Multiplicative Inverse of a non-zero complex number:

Let Z=a+ib be a non zero complex number,

then C-Transformation for Z is= C-T(Z) = $\begin{pmatrix} a & bi \\ bi & a \end{pmatrix}$

Since the multiplication inverse of Z is Z^{-1}

And hence the C-transformation for

$$Z^{-1} = C-T(Z^{-1}) = C-T(Z)^{-1}$$

$$\therefore \begin{pmatrix} a & bi \\ bi & a \end{pmatrix}^{-1}=\frac{\begin{pmatrix} a & -bi \\ bi & a \end{pmatrix}}{\begin{vmatrix} a & -bi \\ bi & a \end{vmatrix}}$$

$$=\frac{\begin{pmatrix} a & -bi \\ bi & a \end{pmatrix}}{(a^2+b^2)}$$

And hence the C-transformation for

$$Z^{-1}=\frac{\begin{pmatrix} a & -bi \\ bi & a \end{pmatrix}}{(a^2+b^2)}$$

And the inverse C-Transformation for

$$\frac{\begin{pmatrix} a & -bi \\ bi & a \end{pmatrix}}{(a^2+b^2)}=\frac{a-ib}{(a^2+b^2)}$$

$$=\frac{a}{(a^2+b^2)}-i\frac{b}{(a^2+b^2)}$$

Hence the Inverse of a+ib $=\dfrac{a-ib}{(a^2+b^2)}$.

Note: By the above definition of addition and multiplication of complex number

If Z= $\cos\theta + i\sin\theta$ then C-T(Z)= $\begin{pmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{pmatrix} = A_\theta$ (say) and

the inverse transformation

$$C-T^{-1}\left(\begin{pmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{pmatrix}\right)$$

$$= C-T^{-1}(A_\theta) = \cos\theta + i\sin\theta = Z$$

Now we shall prove The De-Movier's theorem by applying C-transformation

De-Moiver's Theorem (Special Proof):

Statement: Let 'n' be a positive integer .Then $(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$

Proof : Since C-T($\cos\theta$ +i $\sin\theta$)= C-T(Z)= $\begin{pmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{pmatrix} = A_\theta$ (say)

We know that $A_\theta{}^n = \begin{pmatrix} \cos n\theta & i\sin n\theta \\ i\sin n\theta & \cos n\theta \end{pmatrix}$ for $n \in N$

and the proof as follows

For n=1 , $A_\theta{}^n = \begin{pmatrix} \cos n\theta & i\sin n\theta \\ i\sin n\theta & \cos n\theta \end{pmatrix}$

Let it be true for n=m, so that $(A_\theta)^m = \begin{pmatrix} \cos m\theta & \sin m\theta \\ -\sin m\theta & \cos m\theta \end{pmatrix}$

$$(A_\theta)^{m+1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos m\theta & \sin m\theta \\ -\sin m\theta & \cos m\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta\cos m\theta - \sin\theta\sin m\theta & \cos\theta\sin m\theta + \sin\theta\cos m\theta \\ -\sin\theta\cos m\theta - \cos\theta\sin m\theta & \cos\theta\cos m\theta - \sin\theta\sin m\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos(m+1)\theta & \sin(m+1)\theta \\ -\sin(m+1)\theta & \cos(m+1)\theta \end{pmatrix}$$

Thus the result is true for n=m then it is also true for n= (m+1) .Hence, by the principle of mathematical induction ,it is true for all natural numbers .

$$\therefore \quad A_\theta{}^n = \begin{pmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{pmatrix}$$

And hence Inverse C-Transformation

$$= C - T^{-1}(\ A_\theta{}^n\ )$$

$$= C - T^{-1}(\begin{pmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{pmatrix})$$

$$= (\cos n\theta + i\sin n\theta)$$

And hence $(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$ for $n \in N$.

Cor : If n is a negative integer , then $(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$

Proof: Let n=-p, where p is a positive integer, then $(\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-p}$

$$= \frac{1}{(\cos\theta + i\sin\theta)^p}$$

$$= \frac{1}{(\cos p\theta + i\sin p\theta)}$$

$$= \frac{1}{(\cos p\theta + i\sin p\theta)} \frac{(\cos p\theta - i\sin p\theta)}{(\cos p\theta - i\sin p\theta)} = (\cos p\theta - i\sin p\theta)$$

$$= (\cos(-p)\theta + i\sin(-p)\theta) = (\cos n\theta + i\sin n\theta)$$

Hence the proof.

Note (1): $(\cos\theta + i\sin\theta)^{-n} = (\cos n\theta - i\sin n\theta)$.

Note (2): $(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$.

Note (3): $(\cos\theta - i\sin\theta)^{-n} = (\cos n\theta + i\sin n\theta)$.

Scope: Matrix operation is quite easy to all for solving; I hope this transformation may help for solving the equations involving complex variables. I want to develop this transformation and I wish to apply this in the conformal mappings.

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