EXPERIMENTING WITH THE IDENTITY (XY) Z = Y (ZX)

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Abstract: A non-empty set G together with a binary operation is called a quasi group or groupoid.We knows that a groupoid is a nonempty set with a single binary operation. For a positive integer k, we say that a groupoid is k-nice if the product of any k elements is the same, regardless of their association or order. With this, commutativity is then equivalent to being 2-nice. A groupoid is commutative and associative if and only if it is both 2-nice and 3-nice. In this paper we show that groupoids satisfying identity (xy) z = y (zx) are k-nice for each k $\geq$ 5. Also we see that this yields the corollary that any semiprime ring satisfying (xy)z = y(zx) must be commutative and associative.

Introduction: Now we discuss the results of [2] motivated by experiments with the identity (xy) z = y (zx). Clearly any binary operation that is both commutative and associative will satisfy this identity. While the converse is not true. Using Albert, we notice that with the identity (xy)z = y(zx), a product involving a sufficient number of elements seemed to be independent of the way the elements are ordered or associated.

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Preliminaries: Let G denote a groupoid satisfying (xy) z = y (zx). If a, b, c, d, e$\in$G, there are fourteen ways in which a product on these letters can be associated. We number these as follows.

1. a(b(c(de))) 2. a(b((cd)e))
3. a((bc)(de)) 4. a((b(cd))e)
5. a(((bc)d)e) 6. (ab)(c(de))
7. (ab)((cd)e) 8. (a(bc))(de)
9. ((ab)c)(de) 10. (a(b(cd)))e)
11. (a((bc)d))e 12. ((ab)(cd))e
13. (((ab)c)d)e 14. ((a(bc))d)e

By permuting a,b,c,d,e we can form 1680 =14*5! words. Each word is defined by an association t, 1≤ t ≤14, and a permutation $\pi \in S_5$, the symmetric group on five letters. The permutation π acts on the letters and may be regarded as a renaming. We denote that word by w(t, π). For example, if π = (13) (24) and t = 6, then w(t, π) = (cd) (a(bc)). We goal is now to show that all words in {w(t, π) | 1 ≤ t ≤14, $\pi \in S_5$, } are equal. We observe that for a,b,c,d,$\in$G, by applying (xy)z = y(zx) in different ways we obtain ((ab)c)d = (b(ca))d = (ca)(db) = a((db)c) = a(b(cd)) = ((cd)a)b = (d(ac))b = (ac)(bd) = c((bd)a) = c(d(ab)).

In particular, we have
((ab)c)d = a(b(cd)), ...1
(b(ca))d = (d(ac))b, ...2
a((db)c) = c((bd)a). 3

The last two equations say that for certain associations, the two innermost letters can be interchanged while interchanging the outermost letters.

Next, consider G^1 the groupoid on the same set of elements as G, but with product x o y redefined as yx. It is easy to see that G satisfies (xy)z = y(zx) if and only if G^1 satisfies (xy)z = y(zx). This implies that any identity such as ((ab)c)d = (b(ca))d will derive its mirror image, i.e., d(c(ba)) = d((ac)b). We make heavy use of this technique and will often merely say "by symmetry"

We have also that if w(t, π) = w(t^1, π^1) is an identity, then for any $\sigma \in S_5$, w(t,σ π) = w(t^1,σ π^1) is also an identity.

Lemma 1: for each $\pi \in S_5$,
w (1, π) = w(2, π) = ...= w (14,π).

Proof: By the previous observation, we may assume π is the identity permutation i. Multiplying both sides of equation 1 on the right e, we obtain (((ab)c)d)e = (a(b(cd)))e, and so w (13,i) = w(10,i). 4

Next, we substitute ab for a in 1 and rename b,c,d as c,d,e. We obtain (((ab)c)d)e = (ab)(c(de)), or w(13,i) = w(6,i). 5

Similarly by remaining c,d as d,e in equation 1 and putting bc in place of b, we get w(14,i) = w(3,i). 6

By symmetry, equations 4 - 6 yield respectively,
w(1,i) = w(5,i), ...7 w (1, i) = w(9,i), ...8
w (2, i) = w(12,i). 9

Using $(xy)z = y(zx)$, 3, 9, $(xy)z = y(zx)$, $(xy)z = y(zx)$ and 2, we derive $(((ab)c)d)e = d(e((ab)c)) = d(c((ba)e)) = ((dc)(ba))e = (ba)(e(dc)) = a((e(dc))b) = a((b(cd))e)$ or $w(13,i) = w(4,i)$, ...10

Next, using identity $(xy)z = y(zx)$ repeatedly we obtain

$(a(bc))(de) = ((ca)b)(de) = (e((ca)b))d = ((be)(ca))d = ((a(be))c)d = c(d(a(be))) = c(d((ea)b)) = (((ea)b)c)d = (b(c(ea)))d = (c(ea))(db) = (ea)((db)c) = (ea)(b(cd)) = a((b(cd))e)$, or $w(8,i) = w(4,i)$. 11

Next, $(ab)((cd)e) = b(((cd)e)a) = b(e(a(cd))) = ((a(cd))b)e = (((da)c)b)e = (c(b(da)))e = (b(da))(ec) = (da)((ec)b) = a(((ec)b)d) = a(b(d(ec))) = a(b((cd)e))$, gives $w(7,i) = w(2,i)$. 12

By symmetry, equations 10 - 13 yield respectively,

$w(1,i) = w(11,i)$, ...13

$w(7,i) = w(11,i)$, ...14

$w(8,i) = w(14,i)$. 15

Finally, $((ab)c)(de) = (e((ab)c))d = ((ce)(ab))d = ((b(ce))a)d = a(d(b(ce))) = a(d((eb)c)) = a((cd)(eb)) = a((b(cd))e)$, and we have

$w(9,i) = w(4,i)$. 16

Combining equations .4 - 16 completes the proof.

□

Lemma 2: A groupoid satisfying $(xy)z = y(zx)$ is 5-nice.

Proof: Consider the set $T = \{\sigma \in S_5 \ / \ w(1,i) = w(1,\sigma)\}$. Now using lemma 1, $(xy)z = y(zx)$, and lemma 1 again we obtain $a(b(c(de))) = (ab)((cd)e) = b(((cd)e)a) = b(c(d(ea)))$, or $(12345) \in T$.

Next using identity $(xy)z = y(zx)$ three times and then lemma 1 we obtain $a(b(c(de))) = a(((de)b)c) = a((e(bd))c) = a((bd)(ce)) = a(b(d(ce)))$, or $(34) \in T$. It is well-known [2] that given a cycle of length n and given a transposition, any element in S_n can be expressed as a product involving these two permutations. It is easy to show that T is closed under multiplication. Hence we have $T = S_5$. Finally, let $w(t, \pi)$ be an arbitrary word. By lemma 1, $w(t, \pi) = w(1, \pi)$. Since $\pi \in T$, we have $w(1,\pi) = w(1,i)$.

□

Lemma 3: If $k \geq 3$, a k-nice groupoid is (k+1)-nice.

Proof: we show this holds when k = 5. Passing to the general case will be clear. Let Π represent an arbitrary product involving a_i, $1 \leq i \leq 6$. It suffices to show $\Pi = ((((a_1a_2)a_3)a_4)a_5)a_6$. For some ordering Π can be regarded as a product of the five elements

$(a_{i_1}, a_{i_2}), a_{i_3}, a_{i_4}, a_{i_5}, a_{i_6}$. 17

Case 1: $i_1 \neq 6$, $i_2 \neq 6$. we first apply 5-niceness to the elements in 17, obtaining $\Pi = ((((a_{i_1} a_{i_2}) a_{i_3}) a_{i_4}) a_{i_5}) a_{i_6}$. We then apply it to a_1 through a_5, obtaining $((((a_1a_2)a_3)a_4)a_5)a_6$.

Case 2: $i_1 = 6$. We apply 5-niceness first to the elements in 17, obtaining $\Pi = ((((a_6 a_{i_2}) a_{i_3}) a_{i_4}) a_{i_5}) a_{i_6}$. We then apply 5-niceness to the five leftmost letters, obtaining $((((a_{i_3} a_{i_2}) a_6) a_{i_4}) a_{i_5}) a_{i_6}$. (Note here that we made use of $k \geq 3$.) We now return to case 1.

Case 3: $i_2 = 6$. This is similar to case 2.

Lemma 2 and lemma 3 now yield □ Theorem 1: A groupoid satisfying $(xy)z = y(zx)$ is k-nice for each $k \geq 5$.

Proof: A non associative ring has an additive structure that forms an abelian group, but multiplication is not necessarily commutative or associative. We say a ring R is semiprime if it has no nonzero ideal I for which $I^2 = 0$. Chen theorem 3.2 of [1] showed that a right alternative ring satisfying $(xy)z = y(zx)$, and containing no zero divisors, must be commutative and associative. We now observe that "right alternative" is unnecessary, and "containing no zero divisors" can be replaced by the weaker condition of "semiprime". From here on, let R denote a semiprime ring satisfying $(xy)z = y(zx)$. We first show that R is associative. Let J be the ideal in R generated by all associators, i.e., elements of the form $(uv)w - u(vw)$. It is known that J is the additive span of elements of the form $(uv)w - u(vw)$ and $((uv)w - u(vw))x$. By theorem 1, the product of any two such elements vanishes, and so we have $J^2 = 0$. Since R is semiprime, we must have J = 0. We may now assume R is associative. Next let K be the ideal generated by all commutators, i.e. elements of the form $uv - vu$. By associativity, it is easy to see that K is the additive span of elements of the form $uv-vu$ and $(uv-vu)w$. However by 5-niceness the product of any two elements vanishes with the possible exception of $(uv-vu)(rs-sr)$. But associativity and identity $(xy)z = y(zx)$ imply that any three elements can be cyclicly shifted. Hence $(uv)(rs) = u(vrs) = u(svr) = (usv)r = (svu)r = s(vu)r = (vu)rs = (vu)(rs)$. Similarly $(uv)(sr) = (vu)(sr)$ and so $(uv-vu)(rs-sr) = 0$. It now follows that $K^2 = 0$, and by assumption K= 0, and R is commutative. Thus we have shown □ Corollary 1: A semiprime ring satisfying $(xy)z = y(zx)$ is commutative and associative.

Proof: Therefore with most identities, the number of words increases as the degree increases.

Computer experiments show that other variants of identity, such as $(xy)z = y(xz)$, studied by Thedy [4], or $x(yz) = z(yx)$, studied by Kleinfeld [3] do not have the property of k-niceness, at least at degree 5. We wish to study the property of k-niceness of right alternative rings with $(xy)z = (xz)y$ and $x(yz) = y(xz)$.

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