RINGS WITH ASSOCIATORS IN THE RIGHT NUCLEUS

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Abstract: The associator (x, y, z) is defined by (x, y, z) = (xy) z – x (yz) for all x, y, z in a ring. The commutator (x, y) is defined by (x, y) = xy – yx for all x, y in a ring. This can be considered to be a measure of non-commutativity of ring. By the nucleus N of a ring R, we mean the set of all elements n in R such that (n, R, R) = (R, n, R) = (R, R, n) = 0. A ring R is said to be of characteristic $\neq$ n if n x = 0, impels x = 0 for all $x \in R$ and n is a natural number. A ring R is of characteristic $\neq$ n is simply denoted by char. $\neq$ n. In this paper we prove that if a ring R is simple or prime or semiprime of char. $\neq 2$ with the associators in the right nucleus, then R is associative.

Keywords: Associator, Commutator, Nucleus, Characteristic

Introduction: Kleinfeld [1] proved that if R is a semiprime ring of char. $\neq 2$ satisfying $(R, R, R) \subseteq N_l \cap N_m \cap N_r$, then R is associative. Yen [3] generalized Kleinfeld's result under the weaker hypothesis (R, R, R) contained in two of the three nuclei. In [2] E. Kleinfeld and M. Kleinfeld have shown that, assuming the (R, R, R) in the left nucleus, if R is a simple ring with identity 1 and char. $\neq$ 2, then R must be associative. Without using the identity 1, Yen [4] proved that a simple ring of char. $\neq 2$ with (R, R, R) in the left nucleus is associative. In this paper using the above results we prove that if a ring R is simple or prime or semiprime of char. $\neq 2$, then R is associative.

Preliminaries:

Throughout this paper R represents a nonassociative ring with the associator (R, R, R) in the right nucleus N_r.

i.e., $(R, R, R) \subseteq N_r$. 1

We use the Teichmuller identity which is valid in any arbitrary ring.

$(wx, y,z) - (w,xy,z) + (w,x,yz) = w(x,y,z) + (w,x,y)z$, ...2

for all $w, x, y, z \in R$.

As a consequence of 2, we know that N_l, N_m and N_r are associative subrings of R.

Let $n \in N_r$, then from 2, we obtain

$(x,y,zn) = (x, y, z) n$, ...3

for all $x, y, z \in R$ and $n \in N_r$.

Let I be the associator ideal of a ring R. Then from 2 I can be characterized as all finite sums of associators and right (or left) multiples of associators. Hence, we obtain

$I = (R,R,R) + (R,R,R)R,$

$= (R, R, R) + R(R, R, R)$. 4

First we prove the following lemma.

Lemma 1: Let $T = \{t \in R / tR = 0 = Rt\}$. Then T is an ideal of R.

Proof: Let $t_1, t_2 \in T$. Then for every $x \in R$, we have $t_1x = 0$, $t_2x = 0$, $xt_1 = 0$ and $xt_2 = 0$.

Also $x(t_1 - t_2) = xt_1 - xt_2 = 0$,

$(t_1 - t_2) x = t_1x - t_2x = 0$.

Now for $r \in R$, $t \in T$, both rT and Tr are also in T.

Hence (rt) x = 0 and x (tr) = 0.

i.e., $rt \subseteq T$ and $tr \subseteq T$.

Hence T is an ideal of R.

Theorem 1: Let R be a simple ring and satisfies $(R,R,R) \subseteq N_r$. Then R is associative.

Proof: We assume that R is not associative. Then from 3 and 4, we obtain

$R = R^2 = RI = R(R, R, R)$

$= R^2 = IR = (R, R, R) R$. 5

Using 2 and 1, we obtain

$w(x,y, z) + (w, x,y)z \in N_r$, ...6

for all $w, x, y, z \in R$.

Then with $y \in (R, R, R)$ in 6 and applying 1, we obtain

$w(x,y,z) \in N_r$.

i.e., $R(R,(R,R,R),R) \subseteq N_r$.

Now using this, 1 and 3, we obtain

$0 = (R,R,R(R,(R,R,R),R))$

$= (R,R,R) (R,(R,R,R),R)$. 7

Also $R(R,R,R) \cdot (R,(R,R,R),R) = R \cdot (R,R,R) (R,(R,R,R),R) = 0$ using 7.

Using 5, we have

$R(R,(R,R,R),R) = 0$. 8

From 8 and 5, we have

$0 = R(R,(R,R,R),R)$

$= (R,R,R)R \cdot (R,(R,R,R),R)$

$= (R,I,R)R \cdot (R,(R,R,R),R)$, since R is simple

$= (R,(R,R,R),R)R \cdot (R,(R,R,R),R)$

$= (R,(R,R,R),R)R \cdot (R,I,R)$

$= (R,(R,R,R),R)R \cdot (R,R,R)$

$= (R,(R,R,R),R) \cdot R(R,R,R)$

$= (R,(R,R,R),R) \cdot R$. 9

From 8 and 9, we have

$(R, (R, R, R), R) \subseteq T$. Since T is an ideal of R and R is simple, we have either T = R or T = 0. But T = R implies RR = 0,

a contradiction. Thus T = 0. Hence (R,(R,R,R),R) = 0. Hence the associator

(R, R, R) is now in the middle and the right nucleus.

Now we show that the associator (R, R, R) is also in the left nucleus. For, we replace n by ((a,b,c),d,e) in 3 where a, b, c, d, e $\in$ R and ((a,b,c),d,e) $\in N_r$ from the hypothesis.

Thus, (x,y,z((a,b,c),d,e))=(x,y,z) ((a,b,c),d,e). 10

Now applying 2, we obtain

(x,y,z((a,b,c),d,e))+ (x,y,(z,(a,b,c),d)e) = (x,y,(z(a,b,c),d,e)) − (x,y,(z,(a,b,c)d,e)) + (x,y,(z,(a,b,c),de)). 11

i.e.,

(x,y,z((a,b,c),d,e)) + (x,y,(z,(a,b,c),d)e) = 0 or (x,y,z((a,b,c),d,e)) = − (x,y,(z,(a,b,c),d)e).

Using the fact that associators are in the right and the middle nucleus, we obtain

(x,y,(z,(a,b,c),d)e) = 0.

Thus (x,y,z((a,b,c),d,e)) = 0.

Hence we have,

0 = (R,R,R((R,R,R),R,R))
= (R,R,R) ((R,R,R),R,R))
= I ((R,R,R),R,R). 12

Since I = R, we obtain

R((R,R,R),R,R) = 0. 13

From 13 and 5, we obtain

0 = (R,R,R)R· ((R,R,R),R,R)
= (I,R,R)R· ((R,R,R)R,R,) since R is simple.
= ((R,R,R),R,R)R· (I,R,R)
= ((R,R,R),R,R)· R(R,R,R) since R is simple.
= ((R,R,R),R,R)· R. 14

Thus, from 13 and 14, ((R,R,R),R,R) $\subseteq$ T.

Since T is an ideal of R and R is simple we have either T = R or T = 0. But T = R implies RR = 0, which is again a contradiction. Thus T = 0 and so ((R, R, R), R, R) = 0. Hence we obtain the associators now in the left nucleus as well. Hence (R, R, R) $\subseteq N_l \cap N_m \cap N_r$. Thus from [1] R must be associative. This completes the proof of the theorem.

Corollary 1: Let R be a prime ring and satisfying (R, R, R) $\subseteq N_r$. Then R is associative.

Proof:

Let R be not associative. Then from 7, we have

0 = (R,R,R(R,(R,R,R),R))
= (R,R,R) (R, (R,R,R),R),
= I(R,(R,R,R),R).

From lemma 1 (R,(R,R,R),R) $\subseteq$ T and T is an ideal of R. Hence we obtain I T = 0. But since R is prime we have either I = 0 or T = 0. But I being an associator ideal is not equal to zero. Hence we have T = 0 implying (R, (R, R, R), R) = 0.

i.e., (R, R, R) is in the middle and the right nucleus.

Now from 12 we have,

0 = I((R, R,R),R,R).

But ((R, R, R), R, R) $\subseteq$ T and hence IT = 0. R being a prime ring we have either I = 0 or T = 0. But I is the associator ideal and hence is not equal to zero. Thus we have T = 0 implying ((R, R, R), R, R) = 0.

i.e., (R, R, R) is in the left nucleus as well. Therefore we have (R, R, R) $\subseteq$ N. Now using [1] we obtain a contradiction.

Hence R is associative.

Theorem 2 : If R satisfies the identity (R,R,(R,R,R)) = 0 and if for all a $\in$ R, $a^2 = 0$ implies a = 0, then R must be associative.

Proof : As in theorem 1 we obtain (R,R,R) (R, (R,R,R),R) = 0. Thus $(v,(w,x,y),z)^2 = 0$. Hence (v,(w,x,y),z) = 0 shows that (R,R,R) is in the middle nucleus of R. This leads to (R,R,R)((R,R,R),R,R)=0.Then $((x,y,z),v,w)^2 = 0$. So (R, R, R) is in N, the nucleus of R. Now we use [1] to see that R is associative. This completes the proof of the theorem.

We wish to study some properties of nonassociative rings with (R,R,R) + (R,(R,R,R)) in the right nucleus.

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