"STRONGLY RIGHT ALTERNATIVE RINGS AND BOL CIRCLE LOOPS"

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Abstract: The associator (x, y, z) is defined by (x, y, z) = (xy) z - x (yz) for all x, y, z in R. This plays a central role in the study of non- associative rings. The commutator is defined by (x, y) = xy - yx for all x, y in R. A non-associative ring R is called right alternative if (x, y, y) = 0 for all x, y in R. A groupoid with an identity element is called a loop. A loop is (right) Bol if it satisfies the right Bol identity (xy.z) y = x (yz. y) and moufang if it satisfies the stronger identity (xy.z) y = x (y.zy).In this paper we present some results on strongly right alternative rings which become Bol loops under the circle operation. Also we see that the smallest Bol circle loop has order 16 and there are six such loops.

Keywords: Associator, Non-Associative Ring, Commutator, Right Alternative Ring, Loop.

Introduction: The rings of prime power order are the building blocks of finite rings in general. If R is a finite ring, the abelian group (R, +) is a direct sum of cyclic groups R_p of prime power order and, since $R_p = \{x \in R \,/\, p^n x = 0$ for some $n > 0\}$, it follows readily that R_p is an ideal of R and that the decomposition (R, +) = $\sum_p R_p$ of the additive group of R is actually a decomposition of the ring as the direct product of ideals. If R is right alternative, but not left alternative, then some R_p necessarily has this same property. In [1] Goodaire determined the nilpotent right alternative rings of prime power order p^n, $n \le 4$, which are not left alternative. In this paper we present some results on strongly right alternative rings which become Bol loops under the circle operation. Also we see that the smallest Bol circle loop has order 16 and there are six such loops.

Preliminaries: We know that R is right alternative if

(x, y,y)=0, ...1

for all x, y∈R. By linearization of 1 we have

(x,y,z)+(x,z,y)=0. 2

Right alternative rings satisfies the (right) Bol identity

(xy. z) y = x (yz. y), ...3

[4], an identity which we often wish to have available in 2-divisible as well. Thus, it is convenient to call a ring strongly right alternative if it is a right alternative ring which satisfies the right Bol identity.

A groupoid is a pair (L, .), where L is a set and (a, b) → a . b is a closed binary operation on L. When (L, .) is a groupoid and a∈L, there are maps R(a), L(a):L→L called right and left translations defined by xR(a) = xa, xL(a) = ax, ...4

for x∈L. If these maps are bijections of L for all a∈L, then the groupiod (L, .) is called a quasigroup. Thus, if a . b = c is an equation in a quasigroup, specifying any two of a, b, c uniquely determines the third element. A loop is a quasigroup with a two-sided identity element.

If R is any ring and x, y∈R, define xoy = x + y + xy. An element x∈R is quasi-regular with quasi-inverse x^1 if $xox^1 = x^1ox = 0$. If R is associative and G is the set of quasi-regular elements of R, then (G, o) is a group. If all the elements of R are quasi-regular, (for example, if R is nilpotent), then (G, o) is called the circle group of R. Any abelian group (G, +) is a circle group: define xy = 0 for all x, y∈G. A typical nontrivial result, due to Kaloujinine, says that a p-group G (p an odd prime) which is nilpotent of class 2 is the circle group of a nilpotent ring. The proof involves showing that the operations of multiplication and addition on G defined by $x.\,y = x\,y\,x^{-1}y^{-1}$, $x + y = xy\sqrt{y.x}$ give G the structure of a nilpotent (associative) ring whose circle group is isomorphic to the original group G theorem 1.6.7 of [2]. The author showed that if R is an alternative ring, the quasi-regular elements of R form a Moufang loop under the circle operation [1]. To find classes of Moufang loops which are not associative and which are circle loops, however, is an apparently formidable task. If L is a finite Moufang loop of odd order with commutators in the nucleus, associators in the center and $((x, y), z) = (x, y, z)^2$ for all x, y, z∈L, then the operations of addition and multiplication defined by $x + y = x^{1/2}yx^{1/2}$, $x\,.\,y = (x, y)^{1/2}$, give L the structure of a nilpotent ring whose circle loop is L.

Lemma 1: Let R be a right alternative ring. Then the groupoid (R, o), where xoy = x + y + xy, satisfies the right alternative identity and if R is strongly right alternative, also the right Bol identity.

Proof: By direct computation $(yox)ox - yo(xox) = (yx)x - yx^2$ and $[(xoy)oz]oy - xo[(yoz)oy] = (xy\,.\,z)y - x(yz\,.\,y) + (xy)z + (xz)y - x(yz) - x(zy)$. Since $(xy)z + (xz)y - x(yz) - x(zy) = [x, y, z] + [x, z, y] = 0$ by 2, the result follows.

Next result requires some identities found by Skorniakov [3], which hold in any strongly right alternative ring. Let x, y, z and w denote elements of a strongly right alternative ring R. Using the right Bol identity, we have $[x, yz, y] = (x\,.yz)y - x(yz\,.\,y) = (x\,.\,yz)y - (xy\,.\,z)y = -[x, y, z]y$ and so, in view of 2,

$[x, yz, y] = [x, z, y]y$ 5

Any ring satisfies the Teichmuller identity

$(wx, y, z) - (w, xy, z) + (w, x, yz) = w(x, y, z) + (w, x, y)z$. In R, with w = z, this gives $-[x, yz, z] + [x, y, z^2] = [x, y, z]z = [x, zy, z]$,using 5, and thus

$[x, y, z^2] = [x, yz + zy, z]$. 6

Next, we replace y by y + w in the right Bol identity 3 and obtain $(xy. z) w + (xw. z) y = x(yz . w) + x(wz . y)$. The left side is $[x, y, z]w + (x .yz)w + [x, w, z]y + (x . wz)y$ and so $[x, y, z]w + [x, w, z]y + [x, yz, w] + [x, wz, y] = 0$. Setting w = yz gives $[x, y, z](yz) + [x, yz, z]y + [x, yz^2, y] = 0$. Because of 5 and 6, $[x, yz^2, y] = [x, z^2, y]y = [x, z, yz + zy]y$ and so $0 = [x, y, z](yz) - [x, z, yz]y + [x, z, yz + zy]y = [x,y,z](yz) + [x, z, zy]y$. Since $[x, z, zy] = - [x, zy, z] = - [x, y, z]z$ by 2 and 5, we obtain

$[x, y, z](yz) = \{[x, y, z]z\}y$. 7

Lemma 2: Let R be a strongly right alternative ring, let $x,y \in R$ with x quasi-regular and let x^1 denote the quasi-inverse of x. Then $[y, x, x^1] = 0$.

Proof: First we show that $[y, x, x^1]x^1 = [y, x^1x, x^1]$ because of 5. Since $x + x^1 + x^1x = 0 = [y, x^1, x^1]$, we have $[y, x, x^1]x^1 = -[y, x+x^1,x^1] = -[y, x, x^1]$. From 7, we obtain $[y, x, x^1](xx^1) = \{[y, x, x^1]x^1\}x = -[y, x, x^1]x$. Since $xx^1 = -x-x^1$, then term on the left is $[y, x, x^1](xx^1) = -[y, x, x^1]x - [y, x, x^1]x^1$. Thus $-[y, x, x^1]x^1 = 0$ and $[y, x, x^1] = 0$ as desired. □

Corollary 1: Let R be a strongly right alternative ring. Suppose $a, b \in R$ and a is quasi-regular with quasi-inverse a^1. Then the equation xoa = b has the unique solution $x = boa^1$.

Proof: Since $(ba^1)a = b(a^1a)$, we have $(boa^1)oa = (b + a^1 + ba^1) + a + (b + a^1 + ba^1)a = b + a^1oa + b(a^1oa) = b$. 8

Thus $x = boa^1$ is a solution to xoa = b. A calculation like 8 shows that $(xoa)oa^1 = x$ so, if xoa = b, then $x = boa^1$. The solution is unique.

Lemma 3: Let R be a finite nilpotent strongly right alternative ring. Then (R, o) is a Bol loop.

Proof: By lemma 1, (R, o) satisfies the Bol identity. The groupoid (R, o) has an identity element, namely 0, so it remains only to prove that (R, o) is a quasigroup. For this, we must prove that for any $a \in R$, the left and right translation maps L(a) and R(a) (defined in 4) are bijections. By corollary 1, R(a) is a bijection. Since R is finite, to prove that L(a) is a bijection, it is sufficient to prove that L(a) is one-to-one. Suppose then that $aox_1 = aox_2$ for $a, x_1, x_2 \in R$. Then $x_1 + ax_1 = x_2 + ax_2$, so at = -t with $t = x_1-x_2$. We have a(at) = -at = t and, in general $a(a(.........(at))) = \pm t$. Since R is nilpotent, t = 0. □

Theorem 1: No Bol loop (which is not associative) of order lees than 16 is the circle loop of a nilpotent ring. There are (at least) six Bol loops of order 16 which are not Moufang.

Proof: The first statement follows from corollary 2.2 [1]. For the second statement, we use lemma 3 and the nilpotent rings described in theorems 2.6 and 2.7 of [1] to define Bol loops and provide sufficient information about each loop to make it clear that none is isomorphic to any of the others.

We would like to strengthen lemma 3 and show, as with alternative rings, that the set of quasi-regular elements of any right alternative ring (which also satisfies the right Bol identity) is a Bol loop under the circle operation. Also, using Albert we wish to study the properties of Bol circle loops.

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