WEAKLY NOVIKOV RINGS WITH ASSOCIATORS IN THE NUCLEI

DR. C. JAYA SUBBA REDDY[1] P. GURIVI REDDY[2] D. PRABHAKARA REDDY[3]

Abstract: The associator (x, y, z) is defined by (x, y, z) = (xy) z – x(yz) for all x,y,z in a ring. The commutator (x, y) is defined by (x, y) = xy – yx for all x, y in a ring. This can be considered to be a measure of non-commutativity of ring. By the nucleus N of a ring R, we mean the set of all elements n in R such that (n, R, R) = (R, n, R) = (R, R, n) = 0. A ring R is said to be of characteristic $\neq$ n if n x = 0 impels x = 0 for all $x \in R$ and n is a natural number. In this paper we discuss some results on the rings with the generalized commutators in the nuclei. Here we assume R to satisfy weakly Novikov identity (w, x, yz) = y(w, x, z), for all w,x,y,z in R. We prove that if R is a prime Weakly Novikov ring such that commutator contained in two of the three nuclei, then R is associative or commutative. Also we prove that if R is a semi prime weakly M-ring i.e., (w, xy, z) = x(w, y, z), for all w,x,y,z in R. with $(R, R, R) \subseteq N_m$, then R is associative.

Introduction: Thedy [3] studied rings with commutators in the nuclei. Kleinfeld and Smith [1] studied a class of simple rings with commutators in the left nucleus. They in [2] also described flexible weakly Novikov rings. They proved that, the semiprime flexible weakly Novikov rings are associative. Yen [4] considered prime weakly Novikov ring and semiprime weakly M-ring. He proved that if $T_k \subseteq N_l \cap N_r$ or $T_k \subseteq N_m \cap N_r$ where T_k = (((...((R,R),R)......,R), R), R) then the ring is associative or T_k = 0. In this paper we consider R to be a nonassociative ring satisfying weakly Novikov identity and $T_K = \underbrace{(((....((R,R),R)....,R),R),R)}_{kR's}$

where k is a positive integer. We see that, if R is a semiprime weakly Novikov ring satisfying $(R,R,R) \subseteq N_l$ or $(R,R,R) \subseteq N_r$, then R is associative. Moreover, we prove that if R is a semiprime weakly M-ring satisfying $(R,R,R) \subseteq N_m$, then R is associative.

A ring R is called simple if R is the only nonzero ideal of R. Thus, $R^2 = R$. A ring R is called semiprime if the only ideal of R which squares to zero is the zero ideal. A ring R is called prime if the product of any two nonzero ideals of R is nonzero. If S is a nonempty subset of a ring R, then the ideal of R generated by S is < S >.

Preliminaries:

Thought this paper we consider rings with generalized commutators in the nuclei.

We know that a ring R is weakly Novikov [2] if R satisfies the following identity

(w,x,yz) = y(w,x,z), ...1

for all $w,x,y,z \in R$.

For any ring R,

let $T_K = \underbrace{(((....((R,R),R)....,R),R),R)}_{kR's}$,

where k is a positive integer.

From the above, we have T_2 = (R,R) and T_3 = ((R,R),R). Also $(R,T_k) = (T_k,R) \subseteq T_k$, where k is a positive integer. Obviously we have the following identities.

$T_k + T_k R = T_k + R T_k$ 2

for all positive integers k.

In any arbitrary ring R, we have

S(x,y,z) = (x,y,z) + (y,z,x) + (z,x,y) = (xy,z) + (yz,x)+ (zx,y) 3

for all x,y,z in R.

We shall use the Teichmuller identity

(wx,y,z) – (w,xy,z) + (w,x,yz) = w(x,y,z) + (w,x,y)z 4

for all w,x,y,z in R.

As a consequence of 4, we have that N_l, N_m and N_r are associative subrings of R.

Suppose that $n \in N_l$. Then with w = n in 4, we obtain

(nx,y,z) = n (x,y,z) 5

for all x,y,z in R and $n \in N_l$.

Suppose that $m \in N_r$. Then with z = m in 4, we obtain

(w,x,ym) = (w,x,y)m 6

for all w,x,y in R and m in N_r.

Suppose that $j \in N_l \cap N_m$. Then with x = j in 4, we obtain

(wj,y,z) = (w,jy,z) 7

for all w,y,z in R and j in $N_l \cap N_m$.

Let I be the associator ideal of a ring R. As a consequence of 4 I can be characterized as all finite sums of associators and left multiples of associators. In view of 1 it suffices to take all finite sums of associators if R is a weakly Novikov ring. Hence, in this case I = (R, R, R).

Now we prove the following lemma.

Lemma 1: If R is a weakly Novikov ring, then $RN_r \subseteq N_r$ and $I \cdot N_r = (R, R, R) \cdot N_r = 0$.

Proof: Let $z \in N_r$ and $w,x,y \in R$.

Using 6 and 1, we have

(w,x,y)z = (w,x,yz)
= y (w,x,z)
= 0.

Thus, we get $I \cdot N_r = (R, R, R) \cdot N_r = 0$ and $RN_r \subseteq N_r$.

This completes the proof of the lemma.

For any ring R, let $V_k = T_k + RT_k$ for all positive integers k. In the sequel, for the convenience we denote T_k and V_k by T and V respectively.

Lemma 2: If R is a ring such that T is contained in two of the three nuclei, then V is an ideal of R.

Proof: From 2, we have

$V = T + TR = T + RT.$

Since T is contained in two of the three nuclei, we have $TR \subset V$ and $RT \subset V$.

If T is in the left nucleus, then T+TR is a right ideal, i.e. $VR=TR + T\cdot R^2 \subset V$. If T is in the right nucleus, then T+RT is a left ideal, i.e. $RV=RT +R^2 \cdot T \subset V$.

If T is in the middle nucleus, then V + RV = V + VR because of $RV = R(T+TR) = RT+RT\cdot R \subset V + VR$ and $VR = (T+RT)R = TR+ R\cdot TR \subset V+RV$.

If T is in both the left and right nuclei, then V is a right and left ideal. So V is an ideal. If T is in the middle nucleus, then V + VR = V + RV. So if V is either a right ideal or a left ideal, then V is an ideal. Since T is either in the left or right nucleus, V is a right or a left ideal, and infact V is an ideal. □

Theorem 1: If R is a prime weakly Novikov ring such that $T \subseteq N_l \cap N_r$ or $T \subseteq N_m \cap N_r$, then R associative or T = 0.

Proof: Using $T \subseteq N_r$ and lemma 1, we get

$I\cdot V = I\cdot(T + RT) = 0.$ 8

By lemma 2 and the primeness of R, 8 implies I = 0 or V = 0. Thus, R is associative or T = 0. This completes the proof of the theorem. □

Lemma 3: If R is a weakly Novikov ring, such that $T \subseteq N_l \cap N_m$, then

$(R, R, T) R = 0.$ 9

Proof : We have $(R,T) = (T,R) \subseteq T$. Using this, hypotheses, 4, 1, 7 and 5, for all $y \in T$, and $w,x,z \in R$ we have

$$\begin{aligned}(w,x,y)z &= w(x,y,z) + (w,x,y)z\\ &= (wx,y,z) - (w,xy,z) + (w,x,yz)\\ &= -(w,(x,y),z)-(w,yx,z)+ y(w,x,z)\\ &= -(wy,x,z) + y(w,x,z)\\ &= -((w,y),x,z)-(yw,x,z)+ y(w,x,z)\\ &= 0.\end{aligned}$$

Hence, we get (R, R, T) R = 0.

This completes the proof of the lemma. □

Theorem 2: Let R be a prime weakly Novikov ring such that $T \subseteq N_l \cap N_m$.

If $S(x, y, z) \in N_m$ for all x, y, z in R, or (T, (R, R, R)) = 0, then R is associative or T = 0.

Proof: Assume that $S(x, y, z) \in N_m$ for all x, y, z in R. Using this, 3 and the hypotheses, for all $x \in T$ and $y, z \in R$ we get

$$\begin{aligned}(y,z,x) &= (x,y,z) + (y,z,x) + (z,x,y)\\ &= S(x,y,z) \in N_m.\end{aligned}$$

Thus $(R,R,T) \subseteq N_m$. Applying this, 1 and 9, we have

$$\begin{aligned}(R,R,RT)R &= R(R,R,T)\cdot R\\ &= R\cdot (R,R,T) R\\ &= 0.\end{aligned}$$

Combining the above equation with 9 we have

$(R,R,V)R = 0.$ 10

Assume that (T, (R,R,R)) = 0. Using this, 1, 9 and 4, and noting that $(T,R) \subseteq T$, for all $w, x, y, t \in R$, and $z \in T$ we have

$$\begin{aligned}(w,x,y)z\cdot &= z(w,x,y)\cdot t\\ &= (w,x,zy)t\\ &=(w,x,(z,y))t + (w,x,yz)t\\ &= w(x,y,z)\cdot t + (w,x,y)z\cdot t +\\ &\quad (w,xy,z)t - (wx,y,z)t\\ &= w(x,y,z)\cdot t + (w,x,y)z\cdot t\end{aligned}$$

and $(x,y,wz)t = w(x,y,z)\cdot t = 0$. Combining this with 9, we also obtain 10.

Using 1 and 10, we see that <(R,R,T)> = (R,R,V). By the semiprimeness of R, 10, implies (R,R,V) = 0. By Theorem 1, R is associative or T = 0. This completes the proof of the theorem.

Theorem 3: If R is a prime weakly Novikov ring such that (R,R) is contained in two of the three nuclei, then R is associative or commutative. In the latter case, $N_r = 0$ or R is associative.

Proof: In view of Theorem 1, we may assume that $(R,R) \subseteq N_l \cap N_m$. Let B= (B,R) + R (R,R). By lemma 2, B is an ideal of R. Using lemma 3,

we get $(R,R,(R,R))R = 0.$ 11

Applying 3 and $(R,R) \subseteq N_l \cap N_m$, for all x,y,z in R we have $S(x,y,z) = (x,y,z) + (y,z,x) + (z,x,y) \in N_l \cap N_m$. Let $x \in (R,R)$. Then we get $(y,z,x) \in N_l \cap N_m$. Thus we obtain $(R,R,(R,R)) \subseteq N_l \cap N_m$. Using this and 11, we have $R(R,R,(R,R))\cdot R = R\cdot(R,R,(R,R))R = 0$. Hence, applying this, 1 and 11, and noting that B is an ideal of R, we obtain that $(R,R,B)\cdot R = 0$ and <(R,R,(R,R))> = (R,R,B). Thus, by the semiprimeness of R we get (R, R, B) = 0 and so $(R,R) \subseteq N_r$. By Theorem 1, R is associative or commutative.

Assume that R is commutative. Thus we have $N_r R = R N_r \subseteq N_r$ and $I\cdot N_r = 0$ by lemma 1. Hence N_r is an ideal of R. By the primeness of R, $I\cdot N_r = 0$ implies I = 0 or $N_r = 0$. i.e., $N_r = 0$ or R is associative. This completes the proof of the theorem.

Theorem 4 : If R is a semiprime weakly Novikov ring such that $(R,R,R) \subseteq N_l$ or $(R,R,R) \subseteq N_r$ then R is associative.

Proof: We know that the associator ideal I of R is all finite sums of associators.

Assume that $(R,R,R) \subseteq N_l$. Then by this and 5, for all $w \in (R,R,R)$ and $x,y,z \in R$

we get

$w(x,y,z) = (wx,y,z) \in (I,R,R) = 0.$

Thus, we have (R,R,R) (R,R,R) = 0.

So $I^2 = 0$.

Assume that $(R,R,R) \subseteq N_r$. Then by lemma 1, we obtain

$$(R,R,R)\ (R,R,R) = (R,R,R(R,R,R)) = (R,R,I) = 0.$$

So $I^2 = 0$. In either case, we have $I^2 = 0$. By the semiprimeness of R, this implies I = 0. Thus, R is associative. This completes the proof of the theorem.

We know that a ring R is weakly M-ring [4] if R satisfies the following identity.

$(w,xy,z) = x(w,y,z)$ 12

for all w,x,y,z in R.

If R is a weakly M-ring then by 4 and 12, we have I = (R,R,R).

Now we prove the following theorem.

Theorem 5 : If R is a prime weakly M-ring such that $T \subseteq N_l \cap N_m$ or $T \subseteq N_m \cap N_r$, then R associative or T = 0.

Proof: We have $(T,R) \subseteq T$. Using this, $T \subseteq N_m$ and 12, for all $x \in T$ and $w,y,z,t \in R$ we have

$$\begin{aligned} x(w,y,z) &= x(w,y,z) - y(w,x,z) \\ &= (w,xy,z) - (w,yx,z) \\ &= (w,(x,y),z) \\ &= 0, \end{aligned}$$

and $tx\cdot(w,y,z) = t\cdot x(w,y,z) = 0$.

The above two identities yield

$V\cdot I = 0.$ 13

Since V is an ideal of R, by primeness of R, 13 implies either I = 0 or V = 0. Hence, R is associative or T = 0. This completes the proof of the theorem. □

Theorem 6: If R is a semiprime weakly M-ring such that $(R, R, R) \subseteq N_m$, then R is associative.

Proof: We know that the associator ideal I of R is all finite sums of associators.

Assume that $(R,R,R) \subseteq N_m$.

Since R is weakly M-ring, we have

$x(w,y,z) = (w,xy,z)$ for all w, x, y, z in R. Then by this, for all $x \in (R,R,R)$ and $w, y, z \in R$ we get

$(R,R,R)\ (R,R,R) = (R, (R,R,R)R, R) = (R,I,R) = 0.$

Thus we have $(R,R,R)\ (R,R,R) = 0$. So $I^2 = 0$.

By the semiprimeness of R, this implies I = 0. Thus, R is associative. This completes the proof of the theorem. □

We wish to try for some properties of the weakly M-ring in which the associator is in the left and right nucleus.

References:

1. Kleinfeld, E. and Smith, H.F., "On simple rings with commutators in the left nucleus", Comm. in. Algebra. 19 (1991), 1593-1601.

2. Kleinfeld, E. and Smith, H.F., "Semiprime flexible weakly Novikov rings are associative", Comm. in. Algebra. 23 (13) (1995), 5073-5083

3. Thedy, A., "On rings with commutators in the nuclei", Math. Z., 119 (1971), 213-218.

4. Yen, C.T., "Rings with generalized commutators in the nuclei", Tamkang J. Math. 33 (2002), 371-378.

***

1 Assistant Professor, Department of Mathematics, S.V.University, Tirupathi.
e-mail:cjsreddysvu@gmail.com
2 Lecturer in Mathematics, S.B.V.R Degree College, Badvel, Y.S.R (Dist)..
3 Research Scholar, Department of Mathematics, S.V.University, Tirupathi.