

MV-OPTIMAL AND OTHER OPTIMAL n-ARY DESIGNS

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Abstract : An MV-optimal design minimizes

$$\max_{1 \leq i \leq V} (1/R_{d0} + 1/R_{di})$$

Subject to the restriction $R_{d0} + R_{d1} + \dots + R_{dV} = n$. It is easily seen that an MV-optimal design d^* has

$$R_{d^*i} = R \text{ for } i = 1 \dots V \text{ and}$$

$$R_{d^*0} = n - V \bar{R},$$

Where

$$\hat{R} = \begin{cases} \hat{R}, & \text{if } \hat{R} \text{ is an integer,} \\ [\hat{R}] + 1, & \text{otherwise,} \end{cases}$$

$$\hat{R} = \frac{2n + V - V^2 - (V^4 + V^2 - 2V^3 + 4n^2V)^{1/2}}{2V(V - 1)}$$

We note that for a fixed value of n, the A- and MV-optimality criteria may select substantially different optimal n-ary design from those available. For example, when n=30 and V= 15, an A-optimal design \bar{d} will have $R_{\bar{d}0} = 5$ and $R_{\bar{d}i} = 1$ or 2 for $i = 1, \dots, 15$ whereas the MV-optimal design d^* will have $R_{d^*0} = 15$ and $R_{d^*i} = 1$ for $i = 1, \dots, 15$ and MV- Optimal N-Ary Designs For The One-Way Elimination of Heterogeneity and two way and Robust optimal designs with examples and theorems are given.

Introduction : This Paper recollects the works of Bechhofer and Tamhane (1981) (1983) Majumdar and Notz (1983) Hedayat and Majumdar (1984) Tune (1982,1985) hedayat and Majumdar (1988), Notz (1988) Jacroux (1986, 1987, 1988) Agarwal (1966) and other authors and in particular Hedayat, Mike Jacroux and Dibyen Majumdat (1988) who worked on the outline of the existing knowledge of optimal designs for comparing test treatments with controls under 0,1 and 2way elimination of heterogeneity models. In our Paper we have considered three possible models (i) zero-way elimination of heterogeneity model in which all experimental units are homogeneous before application of treatments, (ii) one-way elimination of heterogeneity model in which experimental units can be divided into several homogeneous blocks and (iii) two-way elimination heterogeneity model in which experimental units can be completely arranged in rows and columns. The results are obtained for A and MV optimal designs for comparing V test treatments with one or many controls under these models stated under n-ary block designs.

Definitions Of Optimal n-Ary Designs

1.1 Definition. Completely Randomized n-ary Design.

$$\theta_1 = \tau_{V \times 1}$$

$$\theta_2 = \mu_{1 \times 1}$$

Given n-ary design d

$X_{id} = ((X_{iid}))_{n \times V}$ observations Vs treatment matrix

1,2...n-1 times the j-th treatment in i-th

observation

$X_{iid} = 0$ otherwise

$X_{2d} = in$

$I_d(\theta_1) = R_d - n^{-1} R_d R_d' (=C_d)$

$R_d = \text{diag}(R_{d1}, R_{d2}, \dots, R_{dV})$

$$R_d = \begin{bmatrix} R_{d1} \\ R_{d2} \\ \cdot \\ \cdot \\ R_{dV} \end{bmatrix}$$

$R_{di} = \#$ times treatment i is represented in d.

1.2 Definition Block n-ary Design

A block n-ary design is an arrangement of V treatment in B blocks each of size $K_{d1}, K_{d2}, \dots, K_{dB}$ respectively. The replication of treatment i is $R_{di}, i = 1, 2, \dots, V$.

$$\theta_1 = \tau_{V \times 1}$$

$$\theta_2 = \begin{bmatrix} \mu \\ \beta \end{bmatrix}_{(B+1) \times 1}$$

$X_{id} = ((x_{ij}))_{n \times V}$ observations vs. treatment matrix

$$X_{2d} = ((I_n X_{\beta d})) (n \times (B+1))$$

$X_{\beta d} = ((x_{\beta id}))$ observations Vs block matrix 1,2... (n-1) if i-th observations occurs in j-th block

$x_{\beta id}$ = otherwise

$$I_d(\theta_1) = R_d - N_d K_d^{-1} N_d' (=C_d)$$

where

$$R_d = \text{diag}(R_{d1} \dots R_{dV})$$

$$K_d = \text{diag}(K_{d1} \dots K_{dB})$$

$N_d = ((n_{dij}))_{(V \times B)}$ treatment-block incidence

n_{dij} is the number of times treatment i appear in the j -th block.

1.3 Definition. Row-Column n-ary Design

A row-column n -ary design d is an arrangement of V treatments in a $(K \times B)$ array of K rows and B columns.

$$\theta_1 = \tau_{V \times 1}$$

$$\theta_2 = \begin{bmatrix} \mu \\ R \\ C \end{bmatrix}_{(K+B+1) \times 1}$$

$$I_d(\theta_1) = R_d - \frac{1}{K} N_d N_d' - \frac{1}{B} M_d M_d' + \frac{1}{BK} R_d R_d' (=C_d)$$

Where

$$R_d = \text{diag}(R_{d1}, R_{d2}, \dots, R_{dV})$$

$N_d = ((n_{dij}))_{(V \times B)}$ treatment-column incidence.

$M_d = ((m_{dij}))_{(V \times K)}$ treatment-row incidence

Optimality criterion to select good n -ary design:-

Suppose $\hat{\eta}_d$ is the BLUE of η using a n -ary

design

$$\text{Var}(\hat{\eta}_d) = V_d$$

It is reasonable to define an optimality criterion for n -ary design as a meaningful function of V_d .

1.4 Definition

A-optimal n -ary design

A n -ary $d^* \in D$ is said to be A-optimal in D iff

$$\text{tr}(V_{d^*}) \leq \text{tr}(V_d)$$

for any other n -ary design $d \in D$

1.5 Definition

D-optimal n -ary design.

A n -ary design $d^* \in D$ is said to be D-optimal in D iff

$$\det(V_{d^*}) \leq \det(V_d)$$

for any n -ary $d \in D$

1.6 Definition

E-optimal n -ary Design

A n -ary $d^* \in D$ is said to be E-optimal in D iff for all normalized treatment constrasts $l' \tau$

with BLUE $l' \hat{\tau}$,

$$\max_{l: l'1=1} (\text{Var}_{d^*}(l' \hat{\tau})) \leq \max_{l: l'1=1} (\text{Var}_d(l' \hat{\tau}))$$

for any other n -ary design $d \in D$.

Let $o = \mu_{d0} < \mu_{d1} \leq \dots \leq \mu_{d, v-1}$ be the eigen values of C_d .

$$\text{Then } \text{Var}_d(l' \hat{\tau}) = \sigma^2 l' C_d^+ l$$

Where C_d^+ is the Moore-Penrose inverse of C_d .

$$\text{Also } \mu_{d, v-1}^{-1} \leq \frac{l' C_d^+ l}{l' 1} \leq \mu_{d1}^{-1}$$

Thus, if $l' 1 = 1$, we have $\max \text{Var}_d(l' \hat{\tau}) \sigma^{-2} = \mu_{d1}^{-1}$.

Hence, a n -ary design $d^* \in D$ is E-optimal in D iff $\mu_{d^*1} \geq \mu_{d1}$ where d is any other competing n -ary design in D (d^* minimizes- (over d)- the maximum variance of all normalized treatment contrasts).

1.7 Definition

MV - optimal n -ary design.

A n -ary design $d^* \in D$ is said to be MV-optimal iff

$$\max_{i \neq j} \text{Var}_{d^*}$$

$$(\hat{\tau}_i, \hat{\tau}_j) \leq \max_{i \neq j} \text{Var}_d(\hat{\tau}_i, \hat{\tau}_j) \text{ Where } d \text{ is any}$$

other competing n -ary design in D

Here our interest is only on elementary treatment contrasts and accordingly the MV-optimality criteria is based on only such specific contrasts.

2. MV - OPTIMAL N - ARY DESIGNS FOR ZERO - WAY ELIMINATION OF HETEROGENEITY

An MV-optimal design minimizes

$$\max_{1 \leq i \leq V} (1/R_{d0} + 1/R_{di})$$

Subject to the restriction $R_{d0} + R_{d1} + \dots + R_{dV} = n$.

It is easily seen that an MV-optimal design d^* has

$$R_{d^*i} = R \text{ for } i = 1 \dots V$$

And

$$R_{d^*0} = n - V \bar{R},$$

Where

$$\hat{R} = \begin{cases} \hat{R}, & \text{if } \hat{R} \text{ is an integer,} \\ [\hat{R}] + 1, & \text{otherwise,} \end{cases}$$

$$\hat{R} = \frac{2n + V - V^2 - (V^4 + V^2 - 2V^3 + 4n^2V)^{1/2}}{2V(V - 1)}$$

We note that for a fixed value of n, the A- and MV-optimality criteria may select substantially different optimal n-ary design from those available. For example, when n=30 and V= 15, an A-optimal design \bar{d} will have $R_{\bar{d}0} = 5$ and $R_{\bar{d}i} = 1$ or 2 for $i = 1, \dots, 15$ whereas the MV-optimal design d^* will have $R_{d^*0} = 15$ and $R_{d^*i} = 1$ for $i = 1, \dots, 15$

3. MV - OPTIMAL N - ARY DESIGNS FOR ONE - WAY ELIMINATION OF HETEROGENEITY

Our n-ary setup is the same as given in earlier. We note that any n-ary design which is A-optimal among all n-ary designs having parameters V,B and K and which estimates all contrasts of the form $t_i - t_0$ with the same variance will also, be MV-optimal. Thus we see that all n-ary designs given in revious chapter VII as being A-optimal are also MV-optimal because all GDTND $(V, B, K, \bar{t}, \bar{s})$'s estimate contrasts of the form $t_i - t_0$ with the same variance. However, Jacroux (1987a) and Soundarapandiyan (1995) have developed some additional sufficient conditions which can be used to establish the MV-optimality of various GDTND (V,B,K,x,z) 's which cannot be proven to be A-optimal using any known results. As an example of the types of results which can be proven for MV-optimality, we have the following.

3.1. Theorem

Using the same notation as introduced in Theorem (7.3.2) ket d^* be a BTNB $(V,B,K, \bar{t}, \bar{s})$ where $n(\bar{t}, \bar{s}) = \min \{ n(x,z); (x,z) \in \hat{\Lambda}^*, (BK-Bx-z)/V \text{ is an integer} \}$ and for positive integers p and q, let $\bar{B}^*(p,q)$ denote the smallest value of y such that

$$(1,-1) \begin{pmatrix} p/K & -y/k \\ -y/k & q/k \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \leq \text{Var}(\hat{t}_{d^*i} - \hat{t}_{d^*0}) \text{ If for any } (x,z) \in \hat{\Lambda}^* \text{ such that } n(x,z) < n(\bar{t}, \bar{s}) \text{ it holds that}$$

$$V \quad K \quad \bar{B}^*(x,z) < (BK-Bx-z-VR^*(x,z)) \times \bar{B}^*(VKB^*(x,z), (R^*(x,z)+1) (K-1) + (V - BK + Bx + z + VR^*(x,z)) \times \bar{B}^*(VKB^*(x,z), R^*(x,z) (K-1)).$$

Then d^* is MV-optimal among all n-ary designs. Using some more complex computational techniques, Jacroux (1987a) and later Soundarapandiyan (1995) have obtained some further results similar to Theorem (8.3.1) which can be used to establish the MV-optimality of various GDTND $(V,B,K, \bar{t}, \bar{s})$'s

having $0 \leq \wedge_2 - \wedge_1 \leq 1$ or $m=V/2, n=2$ and $\wedge_2 = \wedge_1 - 1$ whose A-optimality remains unknown. For example, when $V=6, B=11$ and $K = 5$ as well as when $V=6, B= 16$ and $K = 7$, an A-optimal n-ary design is unknown. However, we are able to give designs n-ary whose MV-optimality can be established using results such as Theorem. These are exhibited below.

3.1 Example.

$V=6, B=11,$ and $K=5$

0	0	0	0	0	0	0	0	0	1	4
1	1	1	2	2	2	3	3	3	2	5
1	1	1	2	2	2	3	3	3	2	5
4	5	6	4	5	6	4	5	6	3	6
4	5	6	4	5	6	4	5	6	3	6

This design is a BTNB $(6,11,5,0,9)$ design.

3.2 Example

$V=6, B=16$ and $K = 7$

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	2	3	1	3	1	2	1	2	1	2	1	3	1	2
1	1	2	3	1	3	1	2	1	2	1	2	1	3	1	2
2	5	4	4	2	5	4	3	3	5	4	3	2	4	3	4
2	5	4	4	2	5	4	3	3	5	4	3	2	4	3	4
3	6	6	5	4	6	5	6	4	6	6	5	5	6	6	5
3	6	6	5	4	6	5	6	4	6	6	5	5	6	6	5

This design is a GDTND $(6,16,7,1,0)$ having treatment groups $(1,4), (2,5)$ and $(3,6)$.

It is interesting to note that the design in Example is an S-type BTNB design, while the design given in Example is a GDTND.

4. MV-OPTIMAL N-ARY DESIGNS FOR THE TWO- WAY ELIMINATION OF HETEROGENEITY

Using arguments similar to those used in previous chapter, we see that all of the A-optimal row-column design which estimate treatment contrasts $t_i - t_0$ with the same variance will also be MV-optimal. Thus all the A-optimal row-column designs listed in chapter VII are also MV-optimal. In addition, a $K \times B$ array is MV-optimal if

- (i) it is an MV-optimal n-ary block designs for 1-way elimination of heterogeneity with columns as blocks, and
- (ii) the total number of replications for each treatment, test treatment or control, is divided equally among the K row.

4.1. Example

For $V=6, B=16,$ and $K=8$ the row-column design given

by
 0 6 4 5 0 3 1 2 0 5 6 2 0 3 1 4
 1 0 6 4 2 0 5 3 1 0 4 3 5 0 6 2
 2 1 0 3 4 6 0 6 3 2 0 5 1 4 0 5
 3 5 2 0 1 5 4 0 4 6 1 0 2 6 3 0
 0 6 4 5 0 3 1 2 0 5 6 2 0 3 1 4
 1 0 6 4 2 0 5 3 1 0 4 3 5 0 6 2
 2 1 0 3 4 6 0 6 3 2 0 5 1 4 0 5
 3 5 2 0 1 5 4 0 4 6 1 0 2 6 3 0

is MV-optimal shown in Jacroux (1986) and Soundarapandiyam (1995).

All of the optimal designs given in this and the preceding chapters possess a high degree of balance in many respects. For example, in terms of the number of replications for test treatments, in terms of the number of joint appearances between the test treatments and the control and between the test treatments themselves in blocks or rows and columns, etc.

Model Robust Optimal n-Ary Designs

In many times the experimenter is not sure whether to fit a one-way or a two-way elimination of heterogeneity model to the data. The performance of several technicians are being compared to the incumbent (control) and the days of the week as well as the hours within each day are the two possible sources of heterogeneity. In such a situation it would be highly desirable to obtain a binary or n-ary design which is A- or MV-optimal under each of these models. Hedayat and Majumdar (1988) and Soundarapandiyam (1995) studied this aspect of the problem and gave some examples of model robust designs. The examples were constructed using the eEuclidean plane, the projective plane and some other geometrical structures. The exact description of the examples are some what involved; some typical results are given below.

5.1 Example

Let $V=4$, $K=3$ and $B=6$. The following design is A- and MV-optimal for both 1- and 2-way elimination of heterogeneity models:

1 0 3 4 2 0
 0 3 4 2 0 1
 4 2 0 0 1 3

In fact, this design is A- and MV-optimal for the o-way elimination of heterogeneity model as well.

5.2 Example

Let $V=7$, $K=8$ and $B=28$. The following n-ary design is A- and MV-optimal for both 1- and 2-way elimination of heterogeneity models

0 0 0 0 0 0 0 1 2 3 4 5 6 7 1 2 3 4 5 6 7 1 2 3 4 5 6 7
 1 2 3 4 5 6 7 0 0 0 0 0 0 0 2 3 4 5 6 7 1 2 3 4 5 6 7 1
 2 3 4 5 6 7 1 2 3 4 5 6 7 1 0 0 0 0 0 0 0 4 5 6 7 1 2 3
 4 5 6 7 1 2 3 4 5 6 7 1 2 3 4 5 6 7 1 2 3 0 0 0 0 0 0 0
 0 0 0 0 0 0 0 1 2 3 4 5 6 7 1 2 3 4 5 6 7 1 2 3 4 5 6 7
 1 2 3 4 5 6 7 0 0 0 0 0 0 0 2 3 4 5 6 7 1 2 3 4 5 6 7 1
 2 3 4 5 6 7 1 2 3 4 5 6 7 1 0 0 0 0 0 0 0 4 5 6 7 1 2 3
 4 5 6 7 1 2 3 4 5 6 7 1 2 3 4 5 6 7 1 2 3 0 0 0 0 0 0 0

We like to mention that the n-ary designs in examples are A-and MV-optimal under zero-way, one-way and two-way elimination of heterogeneity models, whereas the n-ary designs in example (8.3.5) are A-and MV-optimal at least under one-way and two-way elimination of heterogeneity models. We would also like to mention that in our settings, n-ary designs that are known to us to be A- or MV-optimal in the n-way elimination of heterogeneity are also A- or MV-optimal in the (n-1)- way elimination or heterogeneity where $n=2, 1$. However, a general result of this nature has can be proved in the latter research.

Other Optimal n-Ary Designs

We can look at the problem of finding optimal n-ary designs for comparing test treatments with a control, when prior information is available on the parameters of the model. For mathematical tractability, it is convenient to consider this approach in the framework of “continuous” n-ary designs. This means conceptually allowing the R_{di} ’s and the n_{dij} ’s to be real numbers and carrying out the optimization. IN implementing such optimal n-ary designs for the zero-way elimination of heterogeneity model it is necessary to round the nonintegral R_{di} ’s to the nearest integers, keeping the total number of observations in mind; for the one-way elimination of heterogeneity model we have to round the non integral n_{dij} ’s to be nearest integers, keeping the total number of observations in mind.

The research works on Bayes’ A-optimal n-ary designs for zero-way eliminating heterogeneity and Bayes’ A-optimal designs for one-way elimination of heterogeneity runs more than several pages dealing with A-optimal and MV-optimal n-ary designs are not presented here due to space and time which in turn are reserved for future research publications in journals of International reputations.

Conclusion

We present a number of results in the form of definitions, theorems, corollaries and examples, concerning the MV- optimality of n-ary block designs for comparing V number of test treatments with a control for the zero-way, one-way and two-way elimination of heterogeneity. In this paper other efficient n-ary block designs particularly robust optimal n-ary block designs are discussed and research for an approximately A- or MV- optimal n-ary block designs are taken into consideration

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