NEW A-D- AND E-OPTIMAL n-ARY BLOCK DESIGNS WITH UNEQUAL REPLICATIONS AND EQUAL BLOCK SIZES

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Abstract : We shall restrict our attention to connected n-ary block designs i.e. designs whose C-matrices have rank (V-1), and use D (V, B, K) to denote the class of all connected block designs having V treatments arranged in B blocks of size K. For d∈ D (V, B, K) let $O = z_{d0} < z_{d1} \le z_{d2} \le \ldots \le z_{d,V-1}$ denote the non-zero eigen values of C_d.

A n-ary design d* is said to be ϕ_f- optimal in D (V, B, K) if for any other d∈ D (V, B, K)

$$\phi_f\left(C_{di*}\right)=\sum_{r=1}^{V-1} f\left(Z_{di}\right)\le\sum_{r=1}^{V-1} f\left(Z_{di}\right)=\Phi_f\left(C_d\right)$$

where f satisfies the following conditions:

I. f is continuously differentiable on (O,A) where A = max d∈ D (V,B,K) t_r C_d & f' <0, f'' > 0 and f''' <0 on (O,A)

II. f is continuous at 0 or $\lim_{x\to 0} f(x)=\infty$

It is noted that by considering f(x) = 1/x and f(x) = -log(x)

Introduction: Consider D (V:B:K), having v - treatments arranged in B blocks of equal size k, be the class of all connected n-ary block designs . If each block contains the same number of experimental units, these designs are said to be proper n-ary block designs.

Let D (R_1R_2... R_V:B:K) denote the subclasses of n-ary block designs in D (V; B; K) having treatment i replicated R_i times for i=1,2,....V. The criterion considered here is the first E- optimality criterion and then A– and D- optimality criterion. These criterion choose those n-ary block designs in D (V:B:K) and D(R_1, R_2, ... R_V; B; K) having C-matrices whose minimum non-zeros have maximum size. This is equivalent to finding those designs, which has the maximum variance of the estimates of all normalized estimable functions treatment effects.

1. Definition :A n-ary design d* is said to be ϕ_f - optimal in D (V, B, K) if for any other d∈ D (V, B, K)

$$\phi_f\left(C_{di*}\right)=\sum_{r=1}^{V-1} f\left(Z_{di}\right)\le\sum_{r=1}^{V-1} f\left(Z_{di}\right)=\Phi_f\left(C_d\right)$$

where f satisfies the following conditions:

i. f is continuously differentiable on (O,A) where A = max d∈ D (V,B,K) t_r C_d and f' <0, f'' > 0 and f''' <0 on (O,A)

ii. f is continuous at 0 or $\lim_{x\to 0} f(x)=\infty$

It is noted that by considering f(x) = 1/x and f(x) = -log(x), in Definition 3.2 one can obtain the well known A and D-optimality criteria.

Ehrenfeld (1955), Takeuchi (1961, 1963), Kiefer (1975), Cheng (1979), Shafiq (1979) and Jacroux (1980) proved the E-Optimality of various balanced blocks designs in the subclasses of proper designs D (v, b, k) where BK/V = R. Now following definitions of balanced n-ary block design (BNBD) and partially balanced n-ary block (PBNBD) given by Soundrapandian (1980a) and some of the properties of n-ary block design with unequal given by Soundarapandian (1981a), we now investigate the E- A- and D-Optimality of n-ary block designs within classes D (V; B; K) where BK / V is not an integer and in classes D (R_1, R_2, ... R_v, B; K) where Ri are not all equal. Some sufficient conditions are derived for designs to be E- A- and D- optimal within these classes are given.

For a given set of parameters V, B, K and using [x] to denote the integer part of the decimal expansion for x > 0, we shall use the following notation.

R	=	(BK/V)
BK	=	VR + P, O ≤P ≤ V-1
α	=	(BK/BV)
R(x)	=	$(x - B\alpha)(\alpha + 1)^2 + (B - x + B\alpha)\alpha^2$
λ	=	[(RK – R(r)) / (v – 1)]
RK–R(R)	=	(V – 1) λ + q, 0 ≤ q ≤ V – 2.

we consider the determination of A- Optimal designs in various classes D (V, B, K). A good deal of work has already been done on these problem binary designs, Kiefer (1975), Cheng (1978, 1979), Jacroux (1985) and Yeh (1986, 1988). Jacroux (1985) established some numerical sufficient conditions which can often be used to establish ϕ_f-optimality of designs in D(V, B, K) that can be obtained by adding a block to or deleting a block from certain balanced block designs. However, these sufficient conditions do not establish the ϕ_f-optimality of any families of designs and must be applied on a case-by-case basis. Yeh (1988) proved the universal optimality of these same designs over all binary designs in classes D (V, B, K) where K = V – 1. In this chapter and subsequent chapter, we extend the results of Yeh (1988) and establish the A- optimality of certain block designs in classes D (V, B, K) where V-1 or V+1, which can be obtained by adding blocks to and deleting blocks from various balanced n-ary block designs.

Preliminary Results Of N-Ary Block Designs

The binary E- Optimal block designs in the sub classes of proper designs has been characterized by Jacroux(1980a). Now we extend those results to n-ary block designs with unequal replications, with same block sizes. Led $d \in D$ (V; B; K) be V x V positive-semi definite matrix.

$C_d = \text{diag}(R_{d1}, R_{d2}, \ldots R_{dV}) - K^{-1} N_d N'_d$

Here C_d has rank (V – 1) and the normalized eigenvector corresponding to the zero eigenvalues is $\delta_o = (V^{-1/2}, V^{-1/2}, \ldots V^{-1/2})$. The remaining (V – 1) positive eigen values of C_d be

$Z_{d1} \le Z_{d2} \le \ldots \le Z_{dV-1}$ and let T_{xd} be the following matrix.

$K C_d - x (V-1)(VI_v - J_V)$

where x is any real number, I is the (V x V) identify matrix and J_V is the (V x V) matrix.

By calculation, we get the eigen values of T_{xd} as

O, $K_{zdl} - xV/(V-1) \ldots K_{zd\,V-1} - xV/(V-1)$

For a given values of x, t_{xdij} is used to denote the (I,j)-th entry of T_{xd}. Thus,

$$t_{xdii} = R_{di} K - A_{dii} - x; \quad t_{xdij} = A_{dij} + x/(V-1)$$

We also use T_{xdab} to denote the submatrix

$$\begin{vmatrix} t_{xdaa} & t_{xdab} \\ t_{xdab} & t_{xdbb} \end{vmatrix}$$

where T_{xd} and $|T_{xdab}|$ to denote the determinant of this matrix.

Soundarapandian (1981a) established that $B \ge V$ for a balanced n-ary block (BNB) design with V treatments and B blocks. This inequality was shown to be true for a wider class of BNB designs and also for similar balanced n-ary block designs but with same block sizes. The parameter relations given there may be recollected and modified for our present use. The notations for unequally replicated and unequally block sizes of n-ary block designs are given below.

$d \in D (R_1, R_2, \ldots R_V; V; K_1, K_2, \ldots K_B, B; G)$.

Let r_i (i=1, 2, ……, n-1) be the number of blocks in which each element appears i-times. These numbers r_i ($0 \le i \le n-1$) are also constant in all the BNB signs. Similarly let k_j (j=1,2, ….., n-1) be the number of treatments where each treatment is replicated j times.

$$\Delta = \sum_{j=1}^{B} n^2_{ij}; \quad \Lambda = \sum_{j=1}^{B} n_{ij} n_{kj} \quad \text{for all } i \ne k = 1, 2, \ldots V$$

$$\Gamma = \sum_{i=1}^{V} n^2_{ij}; \quad \Pi = \sum_{i=1}^{V} n_{ij} n_{i1} \quad \text{for all } j \ne l = 1, 2, \ldots B$$

$$VR = BK; \qquad V\Delta = B\Gamma$$

$$R = \sum_{i=1}^{n-1} i r_i; \qquad \Lambda(V-1) = \sum_{i=1}^{n-1} i r_i (K-i) = RK - \Delta$$

$\Delta \ge R$; $\Gamma \ge K$

As mentioned above consider a class of connected n-ary block designs D (V; B; G), that p-th treatment has the smallest number of replications Rp fixed and t-th block has the maximum block size K_t fixed. Then

$G_P = VR_P$; $G_t = BK_t$; $G_t \ge G_P$

Where G_P is the total got by each treatment replicated by minimum R_P. G_t is the total got by each block consisting of the maximum size K_t.

Let r_{pj} (j = 1,2, ….. n-1) denote the number of blocks in which each element appears j-times in the p-th treatment replicate and R_P is its total. Similarly, K_{it} (i=1,2,….n-1) represent the number of treatments which replicates i-times in the t-th block and the K_t is the size of the i-th block. Then

$$R_P = \sum_{j=1}^{n-1} p r_{pj}; \qquad \Delta_p = \sum_{j=1}^{n-1} p^2 r_{pj}$$

where R_P is minimum replicate of $R_1, R_2, \ldots R_V$

$$K_t = \sum_{i=1}^{n-1} t k_{it}; \qquad \Gamma_t = \sum_{i=1}^{n-1} t^2 k_{it}$$

where K_t is the maximum block size of $K_1, K_2, \ldots K_B$

Theorems For D ($R_1, R_2, \ldots, R_v$; B; K)

By considering the notation introduced in §, that is $R_P \le R_i$ for i=1, 2, … V, the first extended result for n-ary block design gives the upper and lower bounds for the minimum non zero eigen value of a C-matrix associated with a n-ary block design in D ($R_1, R_2, \ldots R_V$; B ; K)

Theorem : 1.2.1

Let $d \in D (R_1, R_2, \ldots R_V; B; K)$ have C-matrix C_d and let m is the smallest off-diagonal entry occurring in $N_d N'_d = ((\Lambda_{dij}))$. Then

$$mV/K \le Z_{d1} \le (R_p K - \Delta p)V / [(V-1)K]$$

With strict inequality on the right-hand side whenever

$\Lambda_{dpq} \ne (R_{pk} - \Delta_p)/(V-1)$, for some $q \ne p$

Proof: consider T_{xd}, where $x = (R_{pk} - \Delta_p)$. $t_{xdpp} \le 0$ (therefore, $R_p \le \Delta_{dp}$). Since either $t_{xdpp} < 0$ or $t_{xdpp} = 0$ and $|Txdpp| < 0$, we get T_{xd} cannot be positive semi-definite (as $t_{xdpv} \ne 0$ for some $q \ne p$). Here if $t_{xdpq} = 0$ for all $p \ne q$, then rank of T_{xd} is less than (V-1). Thus, T_{xd} must possess a negative eigen value or at least two zero eigen values.

we have $K_{zdl} \le (R_p K - \Delta_p)/(V-1)$ with inequality when $t_{xdq} \ne 0$ for some $q \ne p$ and then we

have the upper bound for Z_{d1}. From $C_d\delta_o = 0$. and $\Delta_{dij} \geq m$ for all $i \neq j$, we get $R_iK - \Delta_d \geq (V-1)m$ for all i.
$t_{xdij} \geq 0$ for i=1, 2, ... V and $t_{xdij} \leq 0$ for $i \neq j$.
Since $T_{xd}\delta_o = 0$, and $x = m(V-1)$, we have

$$\sum_{j\neq 1} |t_{xdij}| = t_{xdij} \quad \text{for } i = 1, 2, \ldots V$$

Thus, T_{xd}, a positive-semidefinite, has non-negative eigen values, we have
$K_{zd1} - mV(V-1)/(V-1) \geq 0$ i.e. $Z_{dl} \geq Vm/K$
Hence the theorem is proved.
Thus if a design exists in D ($R_1, R_2, \ldots R_V$; B; K), whose C-matrix has a minimum nonzero eigen value equal to the upper bound established in Theorem 3.3.1, then that design will clearly be E-Optimal in D ($R_1, R_2, \ldots R_V$; B; K). A sufficient condition for a n-ary block design in D ($R_1, R_2, \ldots R_V$; B; K) to have a C – matrix whose minimum non-zero eigen value is equal to the upper bound established earlier is given below:

Theorem : 1.2.2 Let $d \in D$ ($R_1, R_2, \ldots, R_V$; B; K) have information matrix C_d. If the entries of $N_d N'_d = ((\Lambda_{dij}))$ satisfy the condition that
$\Lambda_{dij} \geq (R_pK-\Delta_p)/(V-1)$ for all $i\neq j$
Then $Z_{d1} = (R_pK-\Delta_p)/\{(V-1)K\}$
and d is E-optimal in D ($R_1, R_2, \ldots, R_V$; B; K)
Proof: Let the entries of $N_dN'_d = ((\Lambda_{dij}))$ satisfy the condition that

$$\Lambda_{dij} \geq (R_pK-\Delta_p)/(V-1) \text{ for all } i\neq j.$$

Let us put $m = (R_pK - \Delta_p)/(V-1)$ in the equation of Theorem then we get $V(R_pK - \Delta_p)/K(V-1) \leq Z_{d1} \leq (R_pK-\Delta_p)V/K(V-1)$ and thus the result follows.

Theorems And Results For D (V;B;K)

Let R and Λ to denote the greatest integers not exceeding BK/V and (RK-Δ) / (V-1) respectively and the incidence N matrix of the n-ary design consists of zero or non zeros in each column. Let V sets of non-negative integers whose sum is BK, contain atleast one element less than or equal to R. Then the sufficient condition for a n-ary block design to be E-Optimal in D (V; B; K) is given below:
Theorem : 1.3.1 Let $d \in D$ (V: B: K) have C-matrix C_d and incidence matrix N_d in the row sums $R_{d1}, R_{d2}, \ldots R_{dV}$. Then $Z_{d1} \leq (RK-\Delta)V/(V-1)K$
Proof : Here we see that $R_{dp} \leq R$. Thus $(R_{dp}K-\Delta_{dp}) \leq (RK - \Delta)$ and the result follows from Theorem. When the minimum non-zero eigen value of C-matrix is equal to the upper bound as in Theorem then the sufficient is given below:
Theorem: 1.3.2
Let $d \in D$ (V; B; K) have C-matrix C_d and incidence matrix N_d. If the entries $N_dN'_d = ((\Delta_{dij}))$ are such that $\Lambda_{dij} \geq (RK - \Delta)/(V-1)$, for all $i \neq j$, then
$Z_{d1} = (RK - \Delta)V/(V-1)K$ and d is E-Optimal in D (V; B; K).

Proof:Let $N_dN'_d$ be the association matrix. As $C_d\delta_o = 0$, we have
$(R_{di}K - \Delta_{di}) \geq (RK - \Delta)$ for i = 1, 2, ... V.
$(R_{dp}K - \Delta_{dp}) \leq (R_{dp}K - \Delta_{dp}) \leq (RK - \Delta)$ and so R_{dp} = R. Then from Theorem we get the result.
For general cases, we have the following theorem, which can be easily proved.
Theorem: 1.3.3 Let $d \in D$ (V; B; K) have C-matrix C_d and let m equal the smallest off-diagonal element of $N_dN'_d$. Now let $d \in D$ (V; B; K) have C-matrix C_d and let m_p denote the greatest integer not exceeding $(R_{dp}K - \Delta_{dp})/(V-1)$. If $m_p < m$, then $Z_{d1} > Z_{d1}$ and d cannot be E-optimal in D (V; B; K).
Corollary:
Let D (V; B; K) where $\Lambda \leq 1$. Now let $m < \Lambda$ be a non-negative integer and let s be the smallest positive integer such that $(RK - \Delta - sk + s)/(V-1) < m+1$. If there exist $d \in D$ (V; B; K) having $N_dN'_d = (\Lambda_{dij})$ where $\Lambda_{dij} \geq m+1$ for all $i \neq j$, then an E-Optimal n-ary block design $d \in D$ (V; B; K) must be such that $R_{di} \geq R - s+1$ for i = 1, 2, ... V.
Corollary:
Suppose there exists a design $d \in D$ (V; B; K) whose association matrix $N_dN'_d = ((\Lambda_{dij}))$ has $\Lambda_{dij} \geq \Lambda$ for all $i \neq j$. If $(RK - \Delta - K+1)/(V-1) < \Lambda$, then an E-Optimal design in D (V; B; K) must have all treatments replicated at least R-times.
Example
Consider the class of design D (V = 5; B = 9; K-5) where =9, and Λ = 6, whose incidence matrix is

$$N_d = \begin{bmatrix} 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \\ 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 \\ 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

It can be verified that there exist n-ary block designs $d \in D$ (5; 9; 5) whose association matrix $N_dN'_d = ((\Lambda_{dij}))$ have $\Lambda_{dij} \geq 6$ for all $i \neq j$. Since the greatest integer not exceeding $(RK - \Delta - K+1)/(V-1)$ is 5, it follows from Corollary that an E-Optimal design is D (5, 9, 5) must have all of its treatments replicated at least 9 times within the 9 blocks.
Example: Consider the class D (7, 7, 7) whose R=7, Λ=6. Let d and d be the ternary designs in D (7, 7, 7) whose incidence matrices are respectively

$$N_d = \begin{bmatrix} 1 & 0 & 0 & 2 & 0 & 2 & 2 \\ 2 & 1 & 0 & 0 & 2 & 0 & 2 \\ 2 & 2 & 1 & 0 & 0 & 2 & 0 \\ 0 & 2 & 2 & 1 & 0 & 0 & 2 \\ 2 & 0 & 2 & 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 2 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 & 2 & 2 & 1 \end{bmatrix}$$

$$N_d = \begin{bmatrix} 2 & 2 & 1 & 0 & 0 & 2 & 0 \\ 0 & 2 & 2 & 1 & 0 & 0 & 2 \\ 2 & 0 & 2 & 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 2 & 2 & 1 & 0 \\ 1 & 0 & 0 & 2 & 0 & 2 & 2 \\ 2 & 1 & 0 & 0 & 2 & 0 & 2 \\ 0 & 0 & 2 & 0 & 2 & 1 & 1 \end{bmatrix}$$

It can be verified that both of these ternary block designs have C-matrices whose minimum non-zero eigen value is 16/3, ie., approximate integer not greater than the value is 5. Both these ternary n-ary block designs are E-Optimal in the class of D (7, 7, 7). However treatment 7 in d is replicated 6 < R = 7 times.

Construction Of E-Optimal N-Ary Block Designs

By augmenting two E-Optimal n-Ary block designs, we get new incidence matrix of E-Optimal new n-ary block design in its class as below:

Theorem: 2.1 Let d ∈ D (R_1, R_2, …… R_V; B; K) and d ∈ D (R_1, R_2, ….. R_V; B; K) both satisfy Theorem within their respective classes. Let treatments in d and d having minimum replications occur in the same rows of N_d and N_d, then the design whose incidence matrix is ($N_d|N_d$) is E-Optimal in D (R_1,R_2,…R_V;B;K) , when R_1 = Ri + Ri for i = 1, 2, … V and B = B + B.

Proof: Let (i, j)-th entries NN'_d and $N_dN'_d$ be Λ_{dij} and Λ_{dij} respectively. Then the (i, j)th entry of (N_d N_d) $(N_dN_d)'$ is $\Lambda_{dij} + \Lambda_{dij}$ like the first two matrices satisfy Theorem our new matrix (N_d N_d) is also E-Optimal n-ary block design in the class D (R_1, R_2,…..R_V; B; K).

Corollary 2.2: Let d ∈ D (R_1, R_2, …. R_V; B; K) be a balanced n-ary block (BNB) design and let d ∈ D (R_1, R_2, ….., R_V; B; K) be another n-ary block design satisfy Theorem, then the new design (N_dN_d) will be E-Optimal in whatever class D (R_1, R_2, …., R_V; B; K) it belongs.

Corollary 2.3: If one design is binary and the other is n-ary block design, then also the augmented incidence matrix be an E-Optimal n-ary block design in that class of many n-ary block designs.

By the above standard augmentation procedure; we can construct infinitely many E-Optimal n-ary block design in that class as given in the following example.

Example 2.4: Let d ∈ D (R_1, R_2, … R_V; B; K) satisfies Theorem and let $R_p \leq R_i$ for i = 1,2 …V. Now let d ∈ D (R_1, R_2,…. R_V; B; K) has R_p = 0. Then the design of n-ary block design (accordingly), whose incidence matrix is (N_d, N_d) is E-Optimal in D (R_1, R_2, …. R_V; B; K) as given above. Since infinitely many n-ary block designs can be obtained from d in this manner, we see that any n-ary block design satisfying Theorem can be used to generate infinitely many E-Optimal n-ary block designs.

The necessary and sufficient condition for the existence of n-ary block designs in D (R_1, R_2,…., R_V; B; K) satisfying Theorem 3.4.2 can be stated as below without proof.

Theorem 2.5: Let D (R_1, R_2, …. R_V; B; K) be non empty. Then a n-ary block design satisfying Theorem 3.4.2 exists in D (R_1, R_2,…. R_V; B; K) if and only if R_p = C (V – 1) for some positive integer C.

Similarly the necessary and sufficient conditions for the existence of a n-ary block designs in D (V; B; K) that satisfy Theorem is given below.

Theorem 2.6:Let D (V; B; K) be non empty. Then a n-ary block design satisfying Theorem, exist in D (V; B; K) if and only if R = C(V-1) for some positive integer C.

The theorems and results for A- and D- Optimal n-ary block designs are presented for unequal replications and unequal block sizes at the end of Chapter IV.

Further many more constructions of A- D – and E optimal n-ary block are available with us, but for want of time and space, we are reserving them for future communication and publication.

Conclusion : Recent developments on optimal block design, A-, D- and E- optimal block designs and latest A- optimal block designs are important research collections in the field of optimal designs. Similarly the review collection of n-ary block designs from Tocher (1952) to recent times, particularly the works of Soundarapandian (1979 a, b, 1980 a, b, c, d,) and Soundarapandian et.al. (1995 a, b, c, d) is a landmark in incomplete block design which have been utilized to find new A-, D- and E- optimal design in our journal. In this journal the properties of E- optimal n-ary block designs with various classes of proper n-ary block designs in which treatments are not replicated the same number of times have been investigated. Several sufficient conditions are derived for n-ary block designs to be E- optimal within the class considered and several methods of constructing n-ary designs, which satisfy these sufficient conditions, are presented. The highlight of this journal is that some results, providing information as how the replications should be assigned to treatments in n-ary E-optimal designs, are presented with illustrations.

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