# DIRECT AND INVERSE LIMITS OF SOME CLASSES OF REGULAR RINGS

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**Abstract:** We prove that the class of strongly regular rings is closed under direct and inverse limits and apply this result to show that many classes of regular rings are also closed under these limits. We also discuss some other rings related to strongly regular rings which are closed under these limits.

**Keywords:** Strongly regular rings, SF-rings, von Neumann regular rings, weakly regular rings.

**Introduction:** It is known that the inverse limit of an inverse system of regular rings may not be regular (see [1], Example 1.3). Goodearl has shown in [2] that the inverse limit of a "surjective" inverse system of regular rings over the set of nonnegative integers is regular (see also the proof in [5]). The aim of this paper is to prove the same result for any inverse system of commutative regular rings, more generally any inverse system of strongly regular rings and some classes of regular rings.

Throughout this paper, R denotes an associative ring with identity, all ring homomorphisms, subrings, modules, module homomorphisms are unitary. An abelian ring is a ring such that every idempotent is central, a reduced ring is a ring without non-zero nilpotent elements. A ring R is symmetric if for $a, b, c$ in R, $abc = 0$ implies $acb = 0$, a left duo ring is a ring such that every left ideal is an ideal, a left quasi-duo ring is a ring such that every maximal left ideal is an ideal. A ring R is (von Neumann) regular if for every a in R, there exists some b in R such that $a = aba$ and R is strongly regular if for every a in R, there exists some b in R such that $a = a^2b$ (see [1], [9]). Following [6], a ring R is left weakly regular if every left ideal of R is idempotent. A left R-module M is p-injective if, for every principal left ideal A of R, every R-homomorphism $A \to M$ can be extended to a R-homomorphism $R \to M$. A ring R is a left V-ring (left p-V-ring) if every simple left R-module is injective (p-injective). Following [7] a ring R is a left SF-ring if every simple left R-module is flat. In this paper we will consider direct limit of a direct system of rings over a directed set only.

We begin with two propositions. They may be known, but we were unable to find a reference in the literature.

Proposition 1. The following classes of rings are closed under the direct and inverse limits:

(a) Abelian rings.
(b) Reduced rings.
(c) Symmetric rings.

Proof. (a) Let $\{R_i, f_{ij}\}$ be a direct system of abelian rings over a directed set I. Let $e \in \varinjlim R_i$ be an idempotent element and a be an arbitrary element of $\varinjlim R_i$. So $e = [x]$ and $a = [y]$ for some elements $x \in R_i$ and $y \in R_j$ for some i and $j \in I$. Now $e^2 = e$ implies that $\exists\, l \in I$ with $i \le l$ such that $(f_{il}(x))^2 = f_{il}(x)$. Let $k \in I$ such that $l \le k, j \le k$. Since $(f_{lk} \circ f_{il})(x) = f_{ik}(x)$ is an idempotent element of $R_k$ and $R_k$ is abelian, we have $f_{ik}(x)f_{jk}(y) = f_{jk}(y)f_{ik}(x)$. This implies that e is a central element. Therefore $\varinjlim R_i$ is an abelian ring.

It follows easily from the definition that any direct product of abelian rings is abelian and a subring of an abelian ring is also abelian. Therefore the inverse limit of any inverse system of abelian rings is again abelian.

(b) Let $\{R_i, f_{ij}\}$ be a direct system of reduced rings over a directed set I. Let $a \in \varinjlim R_i$ be a nilpotent element. Now $a = [x]$ for some element $x \in R_i$, for some $i \in I$ and $a^n = 0, n \ge 1$ implies that $\exists\, j \in I$ with $i \le j$ such that $(f_{ij}(x))^n = 0$. Since $R_j$ is reduced we have $f_{ij}(x) = 0$. Therefore $\varinjlim R_i$ is a reduced ring.

From the definition it follows immediately that the inverse limit of any inverse system of reduced rings is again reduced.

(c) Let $\{R_i, f_{ij}\}$ be a direct system of symmetric rings over a directed set I. Let $a, b, c \in \varinjlim R_i$ such that $abc = 0$ with $a = [x]$, $b = [y]$ and $c = [z]$ for some $x \in R_i$, $y \in R_j$ and $z \in R_k$ where i, j, k are some elements of I. Now $abc = [x][y][z] = 0$ implies that there exists some $l \in I$ with $i, j, k \le l$ such that $f_{il}(x)f_{jl}(y)f_{kl}(z) = 0$ in $R_l$. Since $R_l$ is symmetric, we have $f_{il}(x)f_{kl}(z)f_{jl}(y) = 0$. So $\varinjlim R_i$ is a symmetric ring.

Again it follows from the definition that the inverse limit of any inverse system of symmetric rings is again symmetric.

□

Proposition 2. The following classes of rings are closed under direct limits:

(a) Left weakly regular rings.
(b) Strongly regular rings.
(c) Regular rings.

Proof. (a) Let $\{R_i, f_{ij}\}$ be a direct system of left weakly regular rings over a directed set I and $R = \varinjlim R_i$. To show that R is left weakly regular, it is enough to show that $Ra = (Ra)^2$ for every $a \in R$. Let a be an arbitrary element of R. Now $a = [x]$ for some $x \in R_i$ where i is some element of I. Since $R_i$ is left weakly regular, $x \in (R_i x)^2$. So $x = \Sigma_{r,s\in R_i} rxsx$, a finite sum. Since $[x] = \Sigma_{r,s\in R_i}[rxsx]$ we see that $a \in (Ra)^2$. So $\varinjlim R_i$ is a left weakly regular ring.

(b) Let $\{R_i, f_{ij}\}$ be a direct system of strongly regular rings over a directed set I. If $a \in \varinjlim R_i$ then $a = [x]$ for some element $x \in R_i$ where i is some element of I. Since $R_i$ is strongly regular $\exists\, y \in R_i$ such that $x = x^2 y$. Since $a^2[y] = [x]^2[y] = a$, the result follows.

(c) The proof is the same as that of (b).

□

Though any direct product of regular rings is again regular, a subring of regular ring may not be regular for example $\mathbb{Q}$ is regular but $\mathbb{Z}$ is not regular. Hence we may suspect that the inverse limit of an inverse system of regular rings may not be regular. In fact, Goodearl [1] in Example 1.10 has constructed an example of an inverse system of regular rings whose inverse limit is not regular. However he showed in [2] (see also [5]) that if an inverse system of regular rings over the set of nonnegative integers with usual partial order is surjective , then the inverse limit is also regular. Here we show that the same result holds if we consider any inverse system of commutative regular rings.

Proposition 3. The inverse limit of any inverse system of commutative regular rings is regular.

Proof. Let $\{R_i, f_{ij}\}$ be an inverse system of commutative regular rings over a partially ordered set I and let $a = (a_i)$ be an element of $\varprojlim R_i$. Since $R_i$'s are regular, for each component $a_i$ of a, there exists an element $c_i \in R_i$ such that $a_i = a_i c_i a_i$. Let $b_i = c_i a_i c_i$, then $b_i a_i b_i = (c_i a_i c_i) a_i (c_i a_i c_i) = c_i a_i c_i = b_i$. Also $a_i b_i a_i = a_i (c_i a_i c_i) a_i = a_i c_i a_i = a_i$. Since $a = (a_i) \in \varprojlim R_i$, whenever $i \le j$ we have $f_{ij}(a_j) = a_i$. So whenever $i \le j$, we have $a_i f_{ij}(b_j) a_i = f_{ij}(a_j) f_{ij}(b_j) f_{ij}(a_j) = f_{ij}(a_j) = a_i$, and $f_{ij}(b_j) a_i f_{ij}(b_j) = f_{ij}(b_j) f_{ij}(a_j) f_{ij}(b_j) = f_{ij}(b_j a_j b_j) = f_{ij}(b_j)$.

Now $f_{ij}(b_j) = f_{ij}(b_j)(a_i b_i a_i) f_{ij}(b_j) = b_i f_{ij}(b_j)(a_i f_{ij}(b_j) a_i) = b_i f_{ij}(b_j) a_i = b_i f_{ij}(b_j)(a_i b_i a_i) = b_i (a_i f_{ij}(b_j) a_i) b_i = b_i$, for $i \le j$. Therefore $b = (b_i)$ is an element of $\varprojlim R_i$ and $aba = (a_i)(b_i)(a_i) = (a_i b_i a_i) = (a_i) = a$. Hence $\varprojlim R_i$ is a regular ring.

□

It is well-known that a ring R is regular if and only if every left R-module is flat see [11], Chapter I, Proposition 12.1 and so R is a left SF-ring . In [7] Ramamurthi asked the question whether a left SF-ring is necessarily regular. Although a number of partial results have been obtained the question is still open. But Ramamurthi has shown in the same paper that a commutative ring is SF-ring if and only if it is regular. Also we know that a commutative ring is a V -ring (p-V -ring) if and only if it is regular (see [3], [4]). So we have the following corollary.

Corollary 4. The classes of commutative SF-rings and commutative V-rings (p-V-rings) are closed under direct and inverse limits.

Since commutative regular rings are also strongly regular rings, we prove more generally:

Proposition 5. The inverse limit of any inverse system of strongly regular rings is strongly regular.

Proof. First we record a lemma which is an exercise in Stenström [11] and which follows from lemma 3.3 of Savage [10]. We recall the details for the sake of completeness.

Lemma 6. ([11], Chapter I, Exercise 47(ii)) A ring $R$ is strongly regular if and only if $\forall\, a \in R$, there exists a unique element $b \in R$ such that $a = aba$ and $b = bab$.

Proof. Suppose $R$ is strongly regular, so for each element $a \in R$, $\exists c \in R$ such that $a = aca$. Let $b = cac$. Then $bab = b$ and $aba = a$. Let $d$ be another element of R such that $ada = a$ and $dad = d$. Since $R$ is strongly regular, the relations $ada = a$ and $dad = d$ implies that the idempotent elements $ab, ad$ are central elements and $ab = ba$ , $ad = da$ (see [8]). This implies that $ab = ad$ and so we get $b = d$. Conversely $\forall\, a \in R$, let $b \in R$ be the corresponding unique element such that $a = aba$ and $b = bab$. Write $y = (b + ba - baba)$. Clearly $aya = a$. Let $x = yay$ then we get $axa = a$ and $xax = x$. By uniqueness we have $b = x = yay$. So $ba = baab$ and $a = a^2 b$. □

We continue with the proof of Proposition 5. Let $\{R_i, f_{ij}\}$ be an inverse system of strongly regular rings over a partially ordered set $I$ and $a = (a_i)$ be an element of $\varprojlim R_i$. Since $R_i$'s are strongly regular, by the above lemma, for each component $a_i$ of $a$, there exists a unique element $b_i \in R_i$ such that $a_i = a_i b_i a_i$ and $b_i a_i b_i = b_i$. Since $a = (a_i) \in \varprojlim R_i$, whenever $i \le j$ we have $f_{ij}(a_j) = a_i$. From this relation, whenever $i \le j$, we get that $a_i f_{ij}(b_j) a_i = f_{ij}(a_j) f_{ij}(b_j) f_{ij}(a_j) = f_{ij}(a_i b_i a_i) = f_{ij}(a_j) = a_i$ and $f_{ij}(b_j) a_i f_{ij}(b_j) = f_{ij}(b_j a_j b_j) = f_{ij}(b_j)$.

Since we also have $a_i b_i a_i = a_i$ and $b_i a_i b_i = b_i$, by uniqueness of $b_i$, we get that $f_{ij}(b_j) = b_i$ whenever $i \leq j$. Therefore $b = (b_i)$ is an element of $\varprojlim R_i$. Since $b$ is the unique element of $\varprojlim R_i$ with $aba = a$ and $bab = b$, $\varprojlim R_i$ is a strongly regular ring.

□

We know that the classes of reduced regular rings, abelian regular rings, symmetric regular rings, left duo regular rings and strongly regular rings coincide (see [11], [8]). Therefore Corollary 7 follows immediately.

Corollary 7. The following classes of rings are closed under direct and inverse limits:
(a) Reduced regular rings.
(b) Abelian regular rings.
(c) Symmetric regular rings.
(d) Left duo regular rings.

Contrary to the above proposition, not only the inverse limit of an inverse system of weakly regular rings may not be weakly regular, but the inverse limit of an inverse system of regular rings may not be even weakly regular. The example 1.10 of Goodearl [1] works for this case also. But Rege in [9] has shown that if $R$ is a left quasi-duo ring then the conditions $R$ is a left weakly regular, $R$ is a left V -ring, $R$ is a left p-V -ring, $R$ is a regular ring and $R$ is strongly regular ring are all equivalent. So we have the following corollary.

Corollary 8. The following classes of rings are closed under direct and inverse limits.
(a) Left quasi-duo weakly regular rings:
(b) Left quasi-duo V -rings.
(c) Left quasi-duo p-V -rings.
(d) Left quasi-duo SF-rings.
(e) Left quasi-duo regular rings.

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