SETS OF THE GENERATING FUNCTIONS FOR THE POLYNOMIALS HAVING GENERALIZED APPELL'S REPRESENTATION $A(t)\ \psi(x^2H(t)+xG(t)+F(t))$

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Abstract: This work is motivated by the class of generating functions established earlier by Boas and Buck (1956) and later on the extension given by Rainville (1960). In a recent paper (2003), author has found a natural extension of the above classes which are capable of yielding new generating functions for a large number of new and known polynomial sets. This paper deals with two sets of generating functions obtained for the class of polynomial sets which has generalized Appell's representation of the form $A(t)\ \psi(x^2H(t)+xG(t)+F(t))$. For these sets of generating functions, the recurrence relations and applications have been discussed. Some classes of polynomials suggested by Geganbauer, Hermite and Laguerre polynomials follow as special cases of our findings.

Keywords: Boas and Buck generating functions, Generalized Appell's representation

Introduction: In accordance with Boas and Buck terminology, a set of polynomials has a generalized Appell's representation if it is generated by the formal relation (see, e.g. [2, p.18] and [8, p.72])

$$A(t)\psi(xB(t))=\sum_{n=0}^{\infty}p_n(x)\ t^n\ , \qquad (1)$$

where $p_n(x)$ is a polynomial of degree precisely n if and only if $\gamma_n\neq 0$ given by $\psi(t)=\sum_{n=0}^{\infty}\gamma_n t^n\ ,\ \gamma_0\neq 0$.

Among the polynomial sets included in the Boas and Buck [1, p. 626] classification are:

(i) The Brenke polynomials, which satisfy a generating relation of the type

$$A(t)\psi(xt)=\sum_{n=0}^{\infty}p_n(x)\ t^n\ , \qquad (2)$$

(ii) The Sheffer A-type zero polynomials [8, p. 222] which satisfy the following generating relation

$$A(t)\psi(xB(t))=\sum_{n=0}^{\infty}p_n(x)\ t^n\ , \qquad (3)$$

(iii) The Appell polynomials [7, p.76] which satisfy the generating relation

$$A(t)\exp(xt)=\sum_{n=0}^{\infty}p_n(x)\, t^n\ . \qquad (4)$$

Later on an extension of (1) was studied. This extension is given by the generating relation [8, p.143]

$$A(t)\psi(xH(t)+g(t))=\sum_{n=0}^{\infty}p_n(x)\,t^n\ , \qquad (5)$$

where $p_n(x)$ is a polynomial of degree precisely n if and only if $\gamma_n\neq 0$ given by $\psi(t)=\sum_{n=0}^{\infty}\gamma_n t^n\ ,\ \gamma_0\neq 0$.

The extension of the aforesaid two classes has been established by the author [3] given by the generating relation

$$A(t)\psi(x^2H(t)+xG(t)+F(t))=\sum_{n=0}^{\infty}p_n(x)\,t^n \qquad (6)$$

The recurrence relation for the polynomial $p_n(x)$ and degree of this polynomial are given by the following theorems.

Theorem 1.1 [3] For the polynomial $p_n(x)$ defined by the generating relation

$$A(t)\psi(x^2H(t)+xG(t)+F(t))=\sum_{n=0}^{\infty}p_n(x)\,t^n$$

with $\psi(t)=\sum_{n=0}^{\infty}\gamma_n(x)\,t^n,\ \ \gamma_n\neq 0\quad \forall\ \ n\in\mathbf{N}$.

$$A(t)=\sum_{n=0}^{\infty}a_n(x)\,t^n,\qquad a_0\neq 0 \qquad (8)$$

$$H(t)=\sum_{n=0}^{\infty}h_n(x)\,t^{n+1},\qquad h_0\neq 0 \qquad (9)$$

$$G(t)=\sum_{n=0}^{\infty}g_n(x)\,t^{n+2}, \qquad (10)$$

$$F(t)=\sum_{n=0}^{\infty}f_n(x)\,t^{n+3} \qquad (11)$$

(where G(t) and F(t) are permitted to be identically zero)

holding and $\gamma_n\neq 0$, there exists a sequence of functions of x, $\{\alpha_k\}$, $\{\beta_k\}$, $\{\delta_k\}$, and $\{\rho_k\}$ such that for $n\geq 1$

$$np_n(x)=\sum_{k=0}^{n-2}p'_{n-2-k}(x)\rho_k+\sum_{k=0}^{n}p'_{n-k}(x)\beta_k+\sum_{k=0}^{n-1}\left[p_{n-1-k}(x)\alpha_k+p'_{n-1-k}(x)\delta_k\right] \qquad (12)$$

Indeed

$$\frac{tx^2H'(t)}{2xH(t)+G(t)} = \sum_{n=0}^{\infty}\beta_n(x)t^n, \quad (13)$$

$$\frac{txG'(t)}{2xH(t)+G(t)} = \sum_{n=0}^{\infty}\delta_n(x)t^{n+1}, \quad (14)$$

$$\frac{tx^2F'(t)}{2xH(t)+G(t)} = \sum_{n=0}^{\infty}\rho_n(x)t^{n+2}, \quad (15)$$

$$\frac{tA'}{A}(t) = \sum_{n=0}^{\infty}\alpha_n(x)t^{n+1} \quad (16)$$

Theorem 1.2 [3] If $p_n(x)$ is defined by (7) with (8) to (11) holding, $p_n(x)$ is a polynomial in x and is of degree precisely 2n if and only if $\gamma_n \neq 0$.

Sets Of Generating Functions

In this section, we have defined two sets of generating function which follow from the Theorems 1.1 and 1.2. These sets of generating functions are generalized in nature and lead to a number of generating functions when special values are assigned to the arbitrary constants a,b,c and d.

2.1 First set of generating functions

Consider the set of polynomials $\{p_n(x)\}$ generated by

$$e^{dt}\psi(ax^2t - bxt^2 + ct^3) = \sum_{n=0}^{\infty}p_n(x)t^n \quad (17)$$

then $p'_0(x)=0$ and for $n \geq 1$

$$ax^2p'_n(x) - 2anxp_n(x) = -\{2axd + b(n-1)\}p_{n-1}(x) + dbp_{n-2}(x) + 2bxp'_{n-1}(x) - 3cp'_{n-2}(x) \quad (18)$$

Proof: Let $A(t)=e^{dt}$, $H(t)=at$, $G(t)=-bt^2$ and $F(t)=ct^3$ in (7). Then

$$\frac{tx^2H'(t)}{2xH(t)+G(t)} = \frac{x}{2}\sum_{n=0}^{\infty}\left(\frac{bt}{2ax}\right)^n \Rightarrow \beta_n = \frac{b^nx}{2(2ax)^n}$$

$$\frac{txG'(t)}{2xH(t)+G(t)} = \frac{-bt}{a}\sum_{n=0}^{\infty}\left(\frac{bt}{2ax}\right)^n \Rightarrow \delta_n = \frac{-b^{n+1}}{a(2ax)^n}$$

$$\frac{tx^2F'(t)}{2xH(t)+G(t)} = \frac{3ct^2}{2ax}\sum_{n=0}^{\infty}\left(\frac{bt}{2ax}\right)^n \Rightarrow \rho_n = \frac{3cb^n}{(2ax)^{n+1}}$$

$$\frac{tA'}{A}(t) = td \quad \Rightarrow \alpha_0 = d, \quad \alpha_k = 0 \quad \forall\ k \in \mathbf{N}$$

Substituting the values of α_k, β_k, ρ_k and δ_k in (12), we get

$$np_n(x) = dp_{n-1}(x) + \sum_{k=0}^{n-2}\frac{3cb^k}{(2ax)^{k+1}}p'_{n-2-k}(x) - \sum_{k=0}^{n-1}\frac{b^{k+1}x}{a(2ax)^k}p'_{n-1-k}(x) + \sum_{k=0}^{n}\frac{b^kx}{2(2ax)^k}p'_{n-k}(x)$$

On simplification, we find that

$$np_n(x) = dp_{n-1}(x) - \frac{3b}{4a}p'_{n-1}(x) + \frac{x}{2}p'_n(x) + \frac{3(4ac-b^2)}{4ab}\sum_{k=0}^{n-2}\left(\frac{b}{2ax}\right)^{n-k-1}p'_k(x) \quad (19)$$

Replacing n by (n-1) in above equation and multiplying the result so obtained by $\left(\frac{b}{2ax}\right)$, we get

$$\frac{b(n-1)}{2ax}p_{n-1}(x) = \frac{bd}{2ax}p_{n-1}(x) - \frac{3b^2}{8a^2x}p'_{n-2}(x) + \frac{b}{4a}p'_{n-1}(x) + \frac{3(4ac-b^2)}{4ab}\sum_{k=0}^{n-3}\left(\frac{b}{2ax}\right)^{n-k-1}p'_k(x) \quad (20)$$

Subtracting (20) from (19), we get after a little simplification, the recurrence relation (18).

Now we illustrate how to find $p_n(x)$ using recurrence relation (18). Note that

$$s(2n,n) = a_0\gamma_nh_0^n = a^n\gamma_n \qquad (\because h_0 = 1)$$

where γ_n is given by $\psi(t) = \sum_{n=0}^{\infty}\gamma_n(x)t^n$. For n = 0, we have $\psi_0(x) = s(0,0) = \gamma_0 \quad \Rightarrow \quad \psi'_0(x) = 0$

For n = 1, we have from (18), the differential equation

$$x^2p'_1(x) - 2xp_1(x) = -2xd\,p_{n-1}(x)$$

which on solving, gives us

$$p_1(x) = c_1x^2 + d\gamma_0, \qquad c_1 = s(2,1) = a\gamma_1$$

where c_1 is the constant of integration. Thus

$$p_1(x) = a\gamma_1x^2 + d\,\gamma_0.$$

Similarly for n = 2, we have

$$p'_2(x) - \frac{4}{x}p_n(x) = -\frac{2d^2\gamma_0}{x} - 2ad\gamma_1x + 3b\gamma_1$$

whose solution is given by

$$p_2(x) = c_2x^4 + ad\gamma_1x^2 - b\gamma_1x + \frac{d^2\gamma_0}{2}; \quad c_2 = s(4,2) = a^2\gamma_2$$

Therefore

$$p_2(x) = a^2\gamma_2x^4 + ad\gamma_1x^2 - b\gamma_1x + \frac{d^2\gamma_0}{2}$$

and so on.

2.2 Second set of generating functions

Consider the set of polynomials $\{p_n(x)\}$ generated by

$$e^{dt}\psi\left\{ (x^2t - xt^2 + t^3)e^{at}\right\} = \sum_{n=0}^{\infty} p_n(x)\, t^n \qquad (21)$$

Then $p'_0(x) = 0$ and for $n \geq 1$

$$x^2 p'_n(x) - 2nxp_n(x) = -\{2xd + (n-1)\}p_{n-1}(x) + (2-ax)xp'_{n-1}(x)$$

$$+ dp_{n-2}(x) + \frac{3}{2}(1-4x)p'_{n-2}(x) - 2axp'_{n-3}(x) \quad (22)$$

Proof: Let $A(t) = e^{dt}$, $H(t) = te^{at}$, $G(t) = -t^2 e^{at}$ and $F(t) = t^3 e^{at}$ in (7). Then

$$\frac{tx^2 H'(t)}{2xH(t) + G(t)} = \frac{x}{2}(1+at)\sum_{n=0}^{\infty}\left(\frac{t}{2x}\right)^n,$$

$$\Rightarrow \beta_n = \begin{cases} \dfrac{1+2ax}{4(2x)^{n-1}}, & n \in \mathbf{N} \\ \dfrac{x}{2}, & n = 0 \end{cases}$$

$$\frac{txG'(t)}{2xH(t) + G(t)} = \frac{-t}{2}(2+at)\sum_{n=0}^{\infty}\left(\frac{t}{2x}\right)^n,$$

$$\Rightarrow \delta_n = \begin{cases} \dfrac{-(1+ax)}{(2x)^n}, & n \in \mathbf{N} \\ -1, & n = 0 \end{cases}$$

$$\frac{tx^2 F'(t)}{2xH(t) + G(t)} = t^3(3+at)\sum_{n=0}^{\infty}\left(\frac{t}{2x}\right)^n,$$

$$\Rightarrow \rho_n = \begin{cases} \dfrac{3+2ax}{(2x)^n}, & n \in \mathbf{N} \\ 3, & n = 0 \end{cases}$$

$$\frac{tA'}{A}(t) = td \Rightarrow \alpha_0 = d, \quad \alpha_k = 0 \quad \forall\ k \in \mathbf{N}$$

Substituting the values of α_k, β_k, ρ_k and δ_k from in (12), and simplifying, we get

$$np_n(x) = \frac{(2ax-3)}{4}p'_{n-1}(x) + \frac{x}{2}p'_n(x) + \left(3 - \frac{3+2ax}{8x}\right)p'_{n-2}(x)$$
$$+ (3+2ax)\left(\frac{8x-1}{4}\right)\sum_{k=0}^{n-3}\frac{1}{(2x)^{n-k-1}}p'_k(x) + dp_{n-1}(x) \qquad (23)$$

Replacing n by (n-1) in above equation and multiplying the result so obtained by $1/2x$, we get

$$\frac{(n-1)}{2x}p_{n-1}(x) = \frac{(2ax-3)}{8x}p'_{n-2}(x) + \frac{1}{4}p'_{n-1}(x) + \left(\frac{3}{2x} - \frac{3+2ax}{16x^2}\right)p'_{n-3}(x)$$
$$+ (3+2ax)\left(\frac{8x-1}{4}\right)\sum_{k=0}^{n-4}\frac{1}{(2x)^{n-k-1}}p'_k(x) + \frac{d}{2x}p_{n-1}(x)$$

(24)

Subtracting (24) from (23) and after a little simplification, we obtain the recurrence relation (22).

Further, we note that

$$s(2n,n) = a_0\gamma_n h_0^n = \gamma_n \qquad (\because\ h_0 = 1,\ \ a_0 = 1)$$

where γ_n is given by $\psi(t) = \sum_{n=0}^{\infty}\gamma_n(x)\,t^n$. For n = 0, 1, 2,... we find that

$$p_0(x) = s(0,0) = \gamma_0 \qquad \Rightarrow\ p'_0(x) = 0$$

$$p_1(x) = \gamma_1 x^2 + d\,\gamma_0;$$

$$p_2(x) = \gamma_2 x^4 + (a+d)\gamma_1 x^2 - \gamma_1 x + \frac{d^2\gamma_0}{2}$$

and so on.

Applications

(i) Taking a = 1, b = 2, c = 1, d = 0 in (17) and (18), the set $\{p_n(x)\}$ generated by

$$\psi(x^2t - 2xt^2 + t^3) = \sum_{n=0}^{\infty}p_n(x)\,t^n$$

satisfy the recurrence relation

$$2np_n(x) = xp'_n(x) - 3p'_{n-1}(x), \quad n \geq 1$$

and $p'_0(x) = 0$.

(ii) For a = 3, b = 3, c = 1, d = 0 in (17) and (18) we find that set $\{p_n(x)\}$ generated by

$$\psi(3x^2t - 3xt^2 + t^3) = \sum_{n=0}^{\infty}p_n(x)\,t^n$$

satisfy the recurrence relation

$$x^2p'_n(x) - 2nxp_n(x) = -(n-1)p_{n-1}(x) + 2xp'_{n-1}(x) - p'_{n-2}(x), \quad n \geq 1$$

and $p'_0(x) = 0$.

(27)

This relation was obtained earlier by Khan and Abukhammash [4] by a different technique.

For the different choices of the function $\psi(u)$ here, we can obtain different known sets of polynomials as illustrated below:

(a) For the choice $\psi(u) = e^u$ in (26), we get a new class of polynomials $k_n(x)$, suggested by Hermite polynomials $H_n(x)$, defined by means of generating relation

$$e^{3x^2t - 3xt^2 + t^3} = \sum_{n=0}^{\infty}\frac{k_n(x)}{n!}t^n$$

This set of polynomials was studied earlier by Khan and Abukhammash [4]. The polynomial $k_n(x)$ satisfies the recurrence relation

$$x^2k'_n(x) - 2nxk_n(x)$$
$$= -n(n-1)\{k_{n-1}(x) + k'_{n-2}(x)\} + 2nxk'_{n-1}(x), \ \text{n} \geq 1$$

and $k'_0(x) = 0$. The above relation is obtained by replacing $p_n(x)$ by $\dfrac{k_n(x)}{n!}$ in (27).

(b) Again, if we assume $\psi(u) = (1-u)^{-1/3}$ in (26), we arrive at a different class of polynomials $R_n(x)$, suggested by Legendre polynomials $P_n(x)$ and defined by means of generating relation

$$\left(1-3x^2t+3xt^2-t^3\right)^{-1/3}=\sum_{n=0}^{\infty}R_n(x)\,t^n$$

studied earlier by Khan and Abukhammash [5]. The polynomial $R_n(x)$ satisfies the recurrence relation (27).

(c) If we take $\psi(u)=(1-u)^{-v}$ in (26), we obtain a class of polynomials $A_n^v(x)$, suggested by Geganbauer polynomials $C_n^v(x)$ defined by means of generating relation

$$\left(1-3x^2t+3xt^2-t^3\right)^{-v}=\sum_{n=0}^{\infty}A_n^v(x)\,t^n$$

studied recently by Khan and Khan [6] and which satisfies the recurrence relation (27).

(iii) Letting d = 0 in (21) and (22), we obtain the class of generating functions in which the set $\{p_n(x)\}$ is generated by

$$\psi\{(x^2t-2xt^2+t^3)e^{at}\}=\sum_{n=0}^{\infty}p_n(x)\,t^n$$

satisfy the recurrence relation $p'_0(x)=0$ and

$$x^2p'_n(x)-2nxp_n(x)=-(n-1)p_{n-1}(x)+(2-ax)xp'_{n-1}(x)$$
$$+\frac{3}{2}(1-4x)p'_{n-2}(x)-2axp'_{n-3}(x).$$

Conclusion : The two sets of generating functions obtained here from the Theorem 1.1 and Theorem 1.2 are natural extensions of the class of generating functions defined by Rainville [8]. Theorems 1.1 and 1.2 may be used to find generating functions for the polynomials already existing in literature and find new polynomials from a set of generating functions as has been illustrated in Section 3.

References

1. Boas, R. P. and Buck, R. C., Polynomials defined by generating relations, Amer.Math., 63 (1956), 626-632.
2. Boas, R. P. and Buck, R. C., Polynomial expansion of analytic functions, Springer Verlag, Berlin, 1958.
3. Goyal R., An extension of a class of generating functions, Ultra Science, 15(3)M (2003), 289-292.
4. Khan, M.A. and Abukhammash, G.S., On a new class of polynomial set suggested by Hermite polynomials, Bull. Of Greek Math. Soc., 41(1999), 109-121.
5. Khan, M.A. and Abukhammash, G.S., On a new class of polynomial set suggested by Legendre polynomials, Acta Ciencia Indica Math, 29(4) (2003), 857-65.
6. Khan M. A. and Khan A. R., A study of a new class of polynomial set suggested by Gegenbauer polynomials, Math.Sci.Res. J., 6(9) (2002), 457-472.
7. McBride, E. B., "Obtaining generating functions", Springer Verlag, New York, 1971.
8. Rainville, E. D., "Special Functions", Chelsea Publ. Co., Bronx, New York, 1971. First published by MacMillan, New York, 1960.
9. Sheffer, I.M., "Some properties of polynomial sets of type zero", Duke Math. J.,5 (1939), 590-622, Zbl 0022.01502
10. Khan M. A. and Alidad B., "Polynomial Sets Generated By e^t?(xt)?(xt)", Proyecciones Journal of Mathematics, 29(3) (2010), 201-207.

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