A NEW ITERATIVE ALGORITHM FOR APPROXIMATING COMMON FIXED POINTS FOR ASYMPTOTICALLY NON EXPANSIVE MAPPINGS

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Abstract : The aim of this paper is to prove the new Iterative Algorithm for approximating common fixed points for Asymptotically non expansive mappings . In this paper we have extended some results obtained by different authors.

Keywords : Asymptotically non-expansive mappings, Asymptotically quasi non-expansive mapping ,Banach space, Uniformly convex Banach space.

Introduction : Let K be a nonempty closed convex subset of a real uniformly convex Banach space E. A self-mapping $T : K \to K$ is said to be non-expansive if $\| T(x) - T(y) \| \leq \| x - y \|$ for all $x, y \in K$. A self-mapping $T : K \to K$ is called asymptotically non-expansive if there exist sequences $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ as $n \to \infty$ such that

$$\| T^n(x) - T^n(y) \| \leq k_n \| x - y \|, \forall\, x, y \in K, n \geq 1.$$

A self-mapping $T : K \to K$ is said to be uniformly L-Lipschitzian if there exists constant $L > 0$ such that $\| T^n(x) - T^n(y) \| \leq L \| x - y \|, \forall\, x, y \in K, n \geq 1.$

A self-mapping $T : K \to K$ is called asymptotically quasi-non-expansive if $F(T) \neq \phi$ and there exist sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\| T^n(x) - p \| \leq k_n \| x - p \|, \forall\, x \in K, p \in F(T), n \geq 1.$$

Clearly, if T is an asymptotically non-expansive mapping from K into itself with a fixed point in K, then T is asymptotically quasi-non-expansive, but the converse may be not true.

Definition Let K be a nonempty subset of real-normed linear space E. Let $P : E \to K$ be the non expansive retraction of E onto K.

(1) A non-self-mapping $T : K \to E$ is called asymptotically non expansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\| T(PT)^{n-1}(x) - T(PT)^{n-1}(y) \| \leq k_n \| x - y \|, \forall\, x, y \in K, n \geq 1.$$

(2) T is aid to be uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$\| T(PT)^{n-1}(x) - T(PT)^{n-1}(y) \| \leq L \| x - y \|, \forall\, x, y \in K, n \geq 1.$$

Definition Let K be a nonempty subset of real normed linear space E. Let $P : E \to K$ be a non expansive retraction of E onto K.

(1) A non-self-mapping $T : K \to E$ is called asymptotically non expansive with respect to P if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\| (PT)^n x - (PT)^n y \| \leq k_n \| x - y \|, \quad \forall\, x, y \in K, n \geq 1.$$

(2) T is said to be uniformly L-Lipschitzian with respect to P if there exists a constant $L > 0$ such that

$$\| (PT)^n x - (PT)^n y \| \leq L \| x - y \|, \quad \forall\, x, y \in K, n \geq 1.$$

Lemma: Let $\{\alpha_n\}$ and $\{t_n\}$ be two nonnegative real sequences satisfying

$$\alpha_{n+1} \leq \alpha_n + t_n, \qquad \forall\, n \geq 1.$$

$$\text{If } \sum_{n=1}^{\infty} t_n < \infty, \text{ then } \lim_{n\to\infty} \alpha_n \text{ exists.}$$

Lemma: Let E be a real uniformly convex Banach space and let $B_r(0)$ be the closed ball of E with centre at the origin and radius $r > 0$. Then, there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that $\| \lambda x + \mu y + \gamma z \|^2 \leq \lambda \| x \|^2 + \mu \| y \|^2 + \gamma \| z \|^2 - \lambda\mu g(\| x - y \|)$ for all $x, y, z \in B_r(0)$ and $\lambda, \mu, \gamma \in [0, 1]$ with $\lambda + \mu + \gamma = 1$.

Lemma : Let E be a real smooth and uniformly convex Banach space and K a nonempty closed convex subset of E with P as a sunny non-expansive retraction. Let $T : K \to E$ be a weakly inward and asymptotically non-expansive mapping with respect to P with a sequence $\{k_n\} \subset [1, \infty)$ such that $\{k_n\} \to 1$ as $n \to \infty$. Then $I - T$ is demiclosed at zero, that is, $x_n \to x$ and $x_n - Tx_n \to 0$ imply that $Tx = x$.

Lemma: Let K be a closed convex subset of a uniformly convex Banach space E with a Frechet differentiable norm and let $\{T_n : 1$ □ n □ □$\}$ be a family of Lipschitzian self-mappings of K with a non-empty common fixed point set F and a Lipschitzian constant sequence$\{L_n\}$ such that $\sum_{n=1}^{\infty} (L_n - 1) < \infty$. If $x_1 \in K$ and $x_{n+1} = T_n x_n$ for $n \geq 1$, then $\lim_{n\to\infty} (f_1 - f_2, x_n)$ exists for all $f_1 \neq f_2 \in F$.

Main Results

Theorem Let K be a non-empty closed convex subset of a normed linear space E. Let $T_1, T_2 : K \to E$ be two non-self asymptotically non-expansive mapping with respect to P with sequences $\{k_n\}, \{l_n\} \subset [1, \infty)$; $\sum_{n=1}^{\infty} (k_n - 1) < \infty$; $\sum_{n=1}^{\infty} (l_n - 1) < \infty$, respectively. Suppose that $\{x_n\}$ is the sequence defined by

$$x_{n+1} = \alpha_n (PT_1)^n x_n + \beta_n (PT_2)^n x_n + \gamma_n (PT_3)^n y_n$$

$$y_n = \alpha_n' (PT_1)^n x_n + \beta_n' (PT_2)^n x_n + \gamma_n' (PT_3)^n x_n.$$

If $F(T_1) \cap F(T_2) \cap F(T_3), \neq \phi$, then

$\lim_{n\to\infty} \| x_n - q \|$ and $\lim_{n\to\infty} \| y_n - q \|$ exist for any $q \in F(T_1) \cap F(T_2). \cap F(T_3)$,

Proof : For any $q \in F(T_1) \cap F(T_2) \cap F(T_3)$, using non-expansivity of P

$$\| x_{n+1} - q \| = \| (\alpha_n (PT_1)^n x_n + \beta_n (PT_2)^n x_n + \gamma_n (PT_3)^n y_n) - Pq \|$$

$$\leq \alpha_n h_n \| x_n - q \| + \beta_n k_n \| x_n - q \| + \gamma_n l_n \| y_n - q \| \quad .. \quad .(1)$$

Now $\| y_n - q \| = \| (\alpha_n' (PT_1)^n x_n + \beta_n] (PT_2)^n x_n + \gamma_n' (PT_3)^n x_n) - Pq \|$

$$\leq \alpha_n' h_n \| x_n - q \| + \beta_n' k_n \| x_n - q \| + \gamma_n' l_n \| x_n - q \|$$

$$\| x_{n+1} - q \| \leq \alpha_n h_n \| x_n - q \| + \beta_n k_n \| x_n - q \| + \gamma_n l_n [\alpha_n' h_n \| x_n - q \| + \beta_n' k_n \| x_n - q \| + \gamma_n' l_n \| x_n - q \|$$

$$\leq m_n \| x_n - q \|$$

where $m_n = \max \{h_n, k_n, l_n\}$ for all $n \geq 1$. Clearly $\sum_{n=1}^{\infty} (m_n - 1) < \infty$ as $\{h_n\}, \{k_n\}, \{l_n\} \subset [1, \infty)$ so $\sum_{n=1}^{\infty} (h_n - 1) < \infty$; $\sum_{n=1}^{\infty} (k_n - 1) < \infty$; $\sum_{n=1}^{\infty} (l_n - 1) < \infty$.

Hence $\lim_{n\to\infty} \| x_n - q \|$ exists.

Theorem: Let K be a non-empty closed convex subset of a real uniformly convex Banach space E. Let $T_1, T_2 : K \to E$ be two non-self asymptotically non-expansive mappings with respect to P with sequences $\{k_n\}, \{l_n\} \subset [1, \infty)$, $\sum_{n=1}^{\infty} (k_n - 1) < \infty$; $\sum_{n=1}^{\infty} (l_n - 1) < \infty$; respectively. Suppose that $\{x_n\}$ is sequence defined as theorem where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in $[\in, 1 - \in]$ for some $\in > 0$.

If $F(T_1) \cap F(T_2) \cap F(T_3), \neq \phi$, then

$\| x_n - (PT_1) x_n \| = \| x_n - (PT_2) x_n \| = \| x_n - (PT_3) x_n \| = 0.$

Proof : $\| x_{n+1} - q \|^2 = \| (\alpha_n (PT_1)^n x_n + \beta_n (PT_2)^n x_n + \gamma_n (PT_3)^n y_n) - q \|^2$

$$\leq \| \alpha_n ((PT_1)^n x_n - q) + \beta_n ((PT_2)^n x_n - q) + \gamma_n ((PT_3)^n y_n - (PT_3)^n x_n + (PT_3)^n x_n - q) \|^2$$

$$\leq \| \alpha_n h_n (x_n - q) + \beta_n k_n (x_n - q) + \gamma_n (l_n (y_n - x_n) + (PT_3)^n x_n - q \|^2$$

$$\leq \| \alpha_n h_n (x_n - q) + \beta_n k_n (x_n - q) + \gamma_n ((PT_3)^n x_n - q) \|^2$$

$$\leq \| (\alpha_m + \beta_n)(x_n - q) + \gamma_n ((PT_3)^n x_n - q \|^2$$

$$\leq m_n^2 \| x_n - q \|^2 - \varepsilon^2 g \| x_n - (PT_3)^n x_n \|$$

$$\Rightarrow g (\| x_n - (PT_3)^n x_n \| \to 0 \text{ as } n \to \infty$$

Since $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$.

$$x_n - (PT_3)^n x_n \to 0 \text{ as } n \to \infty$$

Consequently $x_n - (PT_3) x_n \to 0$ as $n \to \infty$

Similarly $x_n - (PT_2) x_n \to 0$ and $x_n - (PT_1) x_n \to 0$.

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