SOME PROPERTIES OF FUZZY SETS ON FINITE TYPE OF KAC-MOODY ALGEBRAS C_4 & D_4

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Abstract : The Kac-Moody algebras has been attracting the attention of a lot of Mathematicians because of its various connections and applications to different branches of Mathematics and Mathematical Physics since the introduction of the subject in 1968, developed simultaneously and independently by Kac and Moody. On the other hand, Fuzzy sets first originated in a seminar paper by Lotfi A.Zadeh in 1965. The theory on fuzzy sets has been applied not only to all branches of Mathematics but also acts as a tool for solving challenging problems in science & technology and social problems.

In this paper, the classical finite type of Kac- Moody algebras C_4 and D_4 are considered. Fuzzy sets are defined on the cartesian product of the root basis of Kac-Moody algebras using an invariant, non degenerate, symmetric bilinear form $< \ . \ , \ . \ >$. Some of the basic properties like support, core, normality, height, cardinality, relative cardinality and convexity are studied; α – level sets and strong α – level sets are computed. α – cut decomposition for these fuzzy sets, associated with C_4 and D_4 families of finite type of Kac-Moody algebras are computed.

Keywords: Core, Convexity, fuzzy set, Generlized Cartan Matrix, height, Kac-Moody algebra, non-degenerate form, normal, root basis, α – cut decomposition, α -level set, strong α -level set..

Introduction:

1.1 Basic definitions of Fuzzy sets

Definition 1[6]: A classical (crisp) set is normally defined as a collection of elements or objects $x \in X$ that can be finite, countable, over countable.

Definition 2[6]: If X is a collection of objects denoted generically by x, then a fuzzy set $\widetilde{A} = \{(x, \mu_{\widetilde{A}}(x))/x \in X\}$ $\mu_{\widetilde{A}}(x)$ is called the “membership function” or “grade of membership” of x in $\widetilde{A}$ that maps X to the membership space M.

Definition 3[6]: The support of a fuzzy set $\widetilde{A}$, $S(\widetilde{A})$ is the crisp set of all $x \in X$ such that $\mu_{\widetilde{A}}(x) > 0$.

Definition 4[6]: The (crisp) set of elements that belong to the fuzzy set $\widetilde{A}$ at least to the degree α is called the α - level set $A_\alpha = \{x \in X / \mu_{\widetilde{A}}(x) \geq \alpha\}$, $A'_\alpha = \{x \in X / \mu_{\widetilde{A}}(x) > \alpha\}$ is called “Strong α –level set” or “Strong α - cut”.

Definition 5[6]: Let $\widetilde{A}$ be a fuzzy set on X. Then the set $\{x \in X / \mu_{\widetilde{A}}(x) = 1\}$ is called the core of the fuzzy set $\widetilde{A}$. This set is denoted by core ($\widetilde{A}$). A fuzzy set $\widetilde{A}$ is said to be normal if $\sup_x \mu_{\widetilde{A}}(x) = 1$

Definition 6[6]: The height of a fuzzy set is the largest membership grade attained by any element in that set.

Definition 7[6]: A fuzzy set $\widetilde{A}$ is convex if $\mu_{\widetilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\mu_{\widetilde{A}}(x_1), \mu_{\widetilde{A}}(x_2)\}$, $\forall$ x_1, x_2 $\in X$ and $\lambda \in [0,1]$. Alternatively, a fuzzy set is convex if all α – level sets are convex.

Definition 8[1]: Let A be a fuzzy set on U and α be a number such that $0 < \alpha \leq 1$. Then by αA we mean a fuzzy set on U, denoted by αA which is such that $(\alpha A)(x) = \alpha A(x)$ for every x in U. This procedure of associating another fuzzy set with the given fuzzy set A is termed as restricted scalar multiplication.

Definition 9[1]: Any fuzzy set A on U can be decomposed as $A = \sup\{\alpha A_\alpha / 0 < \alpha \leq 1\}$. we also write $A = \sum \alpha A_\alpha$ (or) $A = \cup\ \alpha A_\alpha$.

1.2 Basic Definitions on Kac-Moody Algebras

Definition 10[3]: An integer matrix $A = (a_{ij})_{i,j=1}^n$ is a Generalized Cartan Matrix (abbreviated as GCM) if it satisfies the following conditions:

(i) $a_{ii} = 2$ $\forall$ i =1,2,….,n

(ii) $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$ $\forall$ i, j = 1,2,…,n

(iii) $a_{ij} \leq 0$ $\forall$ i, j = 1,2,…,n.

Let us denote the index set of A by N = {1,…,n}. A GCM A is said to decomposable if there exist two non-empty subsets I, J $\subset$ N such that I $\cup$ J = N and $a_{ij} = a_{ji} = 0$ $\forall$ $i \in I$ and $j \in J$. If A is not decomposable, it is said to be indecomposable.

Definition 11[2]: A GCM A is called symmetrizable if DA is symmetric for some diagonal matrix D = diag(q_1,…,q_n), with $q_i > 0$ and q_i's are rational numbers.

Definition 12[2]: A realization of a matrix $A = (a_{ij})_{i,j=1}^n$ is a triple (H, Π, Π^v) where l is the rank of A, H is a 2n - l dimensional complex vector space, $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ and $\Pi^v = \{\alpha_1^v, \ldots, \alpha_n^v\}$ are linearly independent subsets of H* and H respectively, satisfying $\alpha_j(\alpha_i^{\vee}) = a_{ij}$ for i, j = 1,….,n. Π is called the root basis. Elements of Π are called simple roots. The root lattice generated by Π is $Q = \sum_{i=1}^{n} Z\alpha_i$.

Definition 13[2] :The Kac-Moody algebra g(A) associated with a GCM $A = (a_{ij})_{i,j=1}^{n}$ is the Lie algebra generated by the elements e_i , f_i , $i = 1,2,\ldots n$ and H with the following defining relations :

$$[h, h'] = 0, \quad h, h' \in H$$
$$[e_i, f_j] = \delta_{ij}\alpha_i^{\vee}$$
$$[h, e_j] = \alpha_j(h)e_j$$
$$[h, f_j] = -\alpha_j(h)f_j, \quad i, j \in N$$
$$(ade_i)^{1-a_{ij}}e_j = 0$$
$$(adf_i)^{1-a_{ij}}f_j = 0, \quad \forall\ i \neq j,\ i, j \in N$$

The Kac-Moody algebra g(A) has the root space decomposition $g(A) = \oplus_{\alpha \in Q} g_\alpha(A)$ where $g_\alpha(A) = \{x \in g(A) / [h, x] = \alpha(h)x, for\ all\ h \in H\}$. An element α, $\alpha \neq 0$ in Q is called a root if $g_\alpha \neq 0$.

Let $Q = \sum_{i=1}^{n} Z_+\alpha_i$. Q has a partial ordering "$\leq$" defined by $\alpha \leq \beta$ if $\beta - \alpha \in Q_+$, where α , $\beta \in Q$.

Definition 14[2]: For any $\alpha \in Q$ and $\alpha = \sum_{i=1}^{n} k_i\alpha_i$, define support of α , written as supp α , by supp $\alpha = \{i \in N / k_i \neq 0\}$. Let $\Delta (= \Delta(A))$ denote the set of all roots of g(A) and Δ_+ the set of all positive roots of g(A). We have $\Delta_- = -\Delta_+$ and $\Delta = \Delta_+ \cup \Delta_-$.

Proposition 15[2]: A GCM $A = (a_{ij})_{i,j=1}^{n}$ is symmetrizable if and only if there exists an invariant, bilinear, symmetric, non degenerate form on g(A).

Definition 16[2]: To every GCM A is associated a Dynkin diagram S(A) defined as follows: S(A) has n vertices and vertices i and j are connected by max $\{|a_{ij}|, |a_{ji}|\}$ number of lines if $a_{ij} . a_{ji} \leq 4$ and there is an arrow pointing towards i if $|a_{ij}| > 1$. If $a_{ij}. a_{ji} > 4$, i and j are connected by a bold faced edge, equipped with the ordered pair $(|a_{ij}|, |a_{ji}|)$ of integers.

Theorem 17[2]: Let A be a real n x n matrix satisfying (m1), (m2) and (m3).

(m1) A is indecomposable;
(m2) $a_{ij} \leq 0$ for $i \neq j$;
(m3) $a_{ij} = 0$ implies $a_{ji} = 0$

Then one and only one of the following three possibilities holds for both A and ${}^tA$:

(i) det A ≠ 0; there exists u > 0 such that A u > 0; Av ≥ 0 implies v > 0 or v = 0;
(ii) co rank A=1; there exists u > 0 such that Au = 0; Av ≥ 0 implies Av = 0;
(iii) there exists u > 0 such that Au < 0; Av ≥ 0, v ≥ 0 imply v = 0 .

Then A is of finite, affine or indefinite type iff (i), (ii) or (iii) (respectively) is satisfied.

Definition 18[2]: A Kac- Moody algebra g(A) is said to be of finite, affine or indefinite type if the associated GCM A is of finite, affine or indefinite type respectively.

Definition 19[2]: A symmetric bilinear C- valued form $< . , . >$ on a complex Lie algebra g(A) is said to be invariant if ([x, y], z) = (x, [y, z]) for all x, y, z ∈ g(A).

Theorem 20[4]: For any complex n x n matrix A the following statements are equivalent:

a) There exists an invariant, non-degenerate, symmetric, bilinear form C valued form $<\alpha_i, \alpha_j>$ on g(A).

b) A is symmetrizable. Moreover, if these conditions are satisfied, then

c) $< . , . > |\ g_\alpha \times g_\beta$ is

$$= \begin{cases} 0 & \text{if } \alpha, \beta \in \Delta \cup \{0\} \text{ and } \alpha + \beta \neq 0, \\ \text{non degenerately paired,} & \text{if } \alpha, \beta \in \Delta \cup \{0\} \text{ and } \alpha + \beta = 0 \end{cases}$$

In particular, the restriction of $< . , . >$ to H is also non-degenerate.

Note 20[4]: We note that for the finite type of Kac-Moody algebra the rank of the GCM A = n.ie l = n.

In our previous paper [4] we introduced the new concept of fuzzy sets on the root systems of Kac- Moody algebras. The fuzzy set on X=$\pi \times \pi$, where $\pi = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, is defined as follows:

$$\mu_{\tilde{A}}(\alpha_i, \alpha_j) = \frac{|<\alpha_i, \alpha_j>|}{\max|<\alpha_p, \alpha_q>\ |\ p, q = 1,2,\ldots,n} \qquad (1)$$

where <.,.> denotes bilinear, invariant, non-degenerate form defined by $<\alpha_i, \alpha_j>$ = b_{ij}, where B = (b_{ij}) and A = DB.

Then $\tilde{A} = ((\alpha_i, \alpha_j), \mu_{\tilde{A}}(\alpha_i, \alpha_j))$ forms a fuzzy set on $\pi \times \pi$.

Some Fuzzy Properties on the Root Systems of Finite Kac-Moody Algebras: In this section we discuss two members C_4 and D_4 from the classical family. It is to be noted that the GCM associated with

C_4 is symmetrizable whereas the GCM associated with D_4 is symmetric.

Now, consider the classical family C_4, whose Dynkin diagram is,

and the associated GCM is $A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -2 \\ 0 & 0 & -1 & 2 \end{pmatrix}$.

A = D B where $D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -2 \\ 0 & 0 & -2 & 4 \end{pmatrix}$

Let g(A) be the Kac – Moody Lie algebra associated with the GCM A. In the usual notation g(A) denote the Kac-Moody algebra associated with C_4. Let π be the set of simple roots of g(A). Let $\widetilde{A} = ((\alpha_i,\alpha_j), \mu_{\widetilde{A}}(\alpha_i,\alpha_j))$ be the fuzzy set on $\pi \times \pi$ given by equation (1).

The following lemmas describe some properties of this fuzzy set:

Lemma 1: Support of the fuzzy set $\widetilde{A}$ is $\{(\alpha_1,\alpha_1),(\alpha_2,\alpha_2),(\alpha_3,\alpha_3),(\alpha_4,\alpha_4),(\alpha_1,\alpha_2),(\alpha_2,\alpha_1),$ $(\alpha_2,\alpha_3),(\alpha_3,\alpha_2),(\alpha_3,\alpha_4),(\alpha_4,\alpha_3)\}$

Proof: By the definition of support,

Supp($\widetilde{A}$) = $\{(\alpha_i,\alpha_j) \;/\; \mu_{\widetilde{A}}(\alpha_i,\alpha_j) > 0\}$

$= \{(\alpha_1,\alpha_1),(\alpha_2,\alpha_2),(\alpha_3,\alpha_3),(\alpha_4,\alpha_4),(\alpha_1,\alpha_2),(\alpha_2,\alpha_1),$ $(\alpha_2,\alpha_3),(\alpha_3,\alpha_2),(\alpha_3,\alpha_4),(\alpha_4,\alpha_3)\}$

Lemma 2: Height of the fuzzy set $((\alpha_i,\alpha_j), \mu_{\widetilde{A}}(\alpha_i,\alpha_j))$ is 1.

Proof: Height of the fuzzy set is 1 because the maximum membership grade attained is 1.

Lemma 3: Core of the fuzzy set $\widetilde{A}$ is $\{(\alpha_4,\alpha_4)\}$.

Proof: By the definition of core, Core $(\widetilde{A}) = \{(\alpha_i,\alpha_j) / \mu_{\widetilde{A}}(\alpha_i,\alpha_j) = 1\} = \{(\alpha_4,\alpha_4)\}$.

Lemma 4: The fuzzy set $\widetilde{A}$ is normal.

Proof: Since $Sup_{(\alpha_i,\alpha_j)} \mu_{\widetilde{A}}(\alpha_i,\alpha_j) = 1$, the fuzzy set is normal.

Lemma 5: Let A be an indecomposable with the GCM A. Let g(A) be the Kac – Moody Lie algebra associated with C_4. Let $\widetilde{A}$ be the fuzzy set defined on $\pi \times \pi$ given by equation (1). Then $\widetilde{A}$ has the following properties:

(a) The cardinality $|\widetilde{A}|$ = 9/2.

(b) Relative cardinality $||\widetilde{A}||$ = 9 / 32.

Proof: From equation (1), the fuzzy set $\widetilde{A}$ corresponding to the finite type of Kac-Moody algebra associated with C_4 contains 1 element in X having membership grade 1, 5 elements in X having membership grade 1/2 and 4 elements in X having membership grade 1/4.

By the definition of cardinality,

(a) $|\widetilde{A}| = \sum_{x \in X} \mu_{\widetilde{A}}(x) = 9/2$.

(b) $\|\widetilde{A}\| = \frac{|\widetilde{A}|}{|X|} = \frac{9}{32}$.

Theorem 6: Let g(A) be the Kac-Moody Lie algebra associated with the GCM A. Let π be the set of simple roots of g(A). Let $\widetilde{A} = ((\alpha_i,\alpha_j), \mu_{\widetilde{A}}(\alpha_i,\alpha_j))$ be the fuzzy set on $\pi \times \pi$ given by equation (1). Then for the classical algebra C_4, the α - level sets and Strong α - level sets for α = 1, 1/2, 1/3,...,1/k... are given by,

i. $A_1 = \{(\alpha_4,\alpha_4)\}$.

ii. $A_{1/2} = \{(\alpha_1,\alpha_1),(\alpha_2,\alpha_2),(\alpha_3,\alpha_3),(\alpha_4,\alpha_4),(\alpha_3,\alpha_4),(\alpha_4,\alpha_3)\}$, $|A_{1/2}| = 6$.

iii. $A_{1/3} = A_{1/2}$.

iv. $A_{1/4} = \{(\alpha_1,\alpha_1),(\alpha_2,\alpha_2),(\alpha_3,\alpha_3),(\alpha_4,\alpha_4),$ $(\alpha_3,\alpha_4),(\alpha_4,\alpha_3),(\alpha_1,\alpha_2),(\alpha_2,\alpha_1),(\alpha_2,\alpha_3),(\alpha_3,\alpha_2)\}$, $|A_{1/4}| = 10$.

v. $A_{1/5} = A_{1/4}$,

$A_1 \subset A_{1/2} = A_{1/3} \subset A_{1/4} = A_{1/5} = \ldots = A_{1/k} = \ldots$.

$|A_{1/k}| = 10, \; k = 4,5,\ldots$

vi. $A'_{1/2} = \{(\alpha_4,\alpha_4)\}$

vii. $A'_{1/3} = \{(\alpha_1,\alpha_1),(\alpha_2,\alpha_2),(\alpha_3,\alpha_3),(\alpha_4,\alpha_4),$ $(\alpha_3,\alpha_4),(\alpha_4,\alpha_3)\}$, $|A'_{1/3}| = 6$.

viii. $A'_{1/4} = A'_{1/3}$, $|A'_{1/4}| = 6$.

ix. $A'_{1/5} = A'_{1/3} \cup \{(\alpha_1,\alpha_2),(\alpha_2,\alpha_1),(\alpha_2,\alpha_3),(\alpha_3,\alpha_2)\}$, $|A'_{1/5}| = 10$.

x. $A'_{1/6} = A'_{1/5}$,

$A'_{1/2} \subset A'_{1/3} = A'_{1/4} \subset A'_{1/5} = A'_{1/6} = \ldots = A'_{1/k} = \ldots$,

$|A'_{1/k}| = 10, \; k = 5,6,\ldots$

Proof: i. To compute 1- level Set:

$A_1 = \{ (\alpha_i,\alpha_j) \in X / \mu_{\widetilde{A}}(\alpha_i,\alpha_j) \geq 1 \}$

$$= \left\{ (\alpha_i,\alpha_j) \in X / \frac{|<\alpha_i,\alpha_j>|}{\max|<\alpha_p,\alpha_q>| \; p,q = 1,2,3} \geq 1 \right\} = \{(\alpha_4,\alpha_4)\}.$$

ii. To compute 1/2 –level Set:

$A_{1/2} = \{ (\alpha_i,\alpha_j) \in X / \mu_{\widetilde{A}}(\alpha_i,\alpha_j) \geq 1/2 \}$

$$= \left\{ (\alpha_i,\alpha_j) \in X / \frac{|<\alpha_i,\alpha_j>|}{\max|<\alpha_p,\alpha_q>| \; p,q = 1,2,3} \geq 1/2 \right\}$$

$= \{(\alpha_1,\alpha_1),(\alpha_2,\alpha_2),(\alpha_3,\alpha_3),(\alpha_4,\alpha_4),(\alpha_3,\alpha_4),(\alpha_4,\alpha_3)\}$, $|A_{1/2}| = 6$.

iii. $A_{1/3} = \{(\alpha_i,\alpha_j) \in X / \mu_{\widetilde{A}}(\alpha_i,\alpha_j) \geq 1/3\}$

$= \{(\alpha_1,\alpha_1),(\alpha_2,\alpha_2),(\alpha_3,\alpha_3),(\alpha_4,\alpha_4),(\alpha_3,\alpha_4),(\alpha_4,\alpha_3)\} = A_{1/2}$, $|A_{1/3}| = 6$.

iv. $A_{1/4}=\{(\alpha_i,\alpha_j)\in X/\mu_{\widetilde{A}}(\alpha_i,\alpha_j)\ge 1/4\}$

$=A_{1/3}\cup\{(\alpha_1,\alpha_2),(\alpha_2,\alpha_1),(\alpha_2,\alpha_3),(\alpha_3,\alpha_2)\},\ |A_{1/4}|=10.$

v. $A_{1/5}=\{(\alpha_i,\alpha_j)\in X/\mu_{\widetilde{A}}(\alpha_i,\alpha_j)\ge 1/5\}=A_{1/4}$

From the above results

$A_1\subset A_{1/2}=A_{1/3}\subset A_{1/4}=\ldots A_{1/k}=\ldots,\ |A_{1/k}|=10,\ k=4,5,\ldots$

vi. $A'_{1/2}=\{(\alpha_i,\alpha_j)\in X/\mu_{\widetilde{A}}(\alpha_i,\alpha_j)\ >\ 1/2\}\ =\{(\alpha_4,\alpha_4)\}$

vii. $A'_{1/3}=\{(\alpha_i,\alpha_j)\in X/\mu_{\widetilde{A}}(\alpha_i,\alpha_j)\ >\ 1/3\}$

$=\{(\alpha_1,\alpha_1),(\alpha_2,\alpha_2),(\alpha_3,\alpha_3),(\alpha_4,\alpha_4),(\alpha_3,\alpha_4),(\alpha_4,\alpha_3)\},$

$|A'_{1/3}|=6.$

viii. $A'_{1/4}=\{(\alpha_i,\alpha_j)\in X/\mu_{\widetilde{A}}(\alpha_i,\alpha_j)>1/4\}$

$=\{(\alpha_1,\alpha_1),(\alpha_2,\alpha_2),(\alpha_3,\alpha_3),(\alpha_4,\alpha_4),(\alpha_3,\alpha_4),(\alpha_4,\alpha_3)\}=A'_{1/3},$

$|A'_{1/4}|=6.$

ix. $A'_{1/5}=\{(\alpha_i,\alpha_j)\in X/\mu_{\widetilde{A}}(\alpha_i,\alpha_j)>1/5\}$

$=A'_{1/3}\cup\{(\alpha_1,\alpha_2),(\alpha_2,\alpha_1),(\alpha_2,\alpha_3),(\alpha_3,\alpha_2)\},\ |A'_{1/5}|=10.$

x. $A'_{1/6}=\{(\alpha_i,\alpha_j)/\mu_{\widetilde{A}}(\alpha_i,\alpha_j)>1/6\}=A'_{1/5}$

From the above results,

$A'_{1/2}\subset A'_{1/3}=A'_{1/4}\subset A'_{1/5}=A'_{1/6}=\ldots A'_{1/k}=\ldots,\ |A'_{1/k}|=10,\ k=5,6,\ldots$

Lemma 7: A fuzzy set $\widetilde{A}$ corresponding to the finite type of classical family C_4, defined by equation (1) is convex.

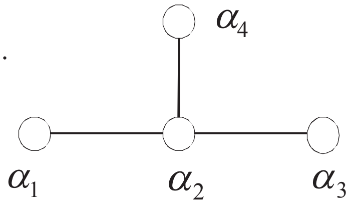
Proof: Consider the finite type of classical family C_4. The table showing all possible membership grades for the elements of X and the conditions for checking convexity is listed below:

$\mu_{\widetilde{A}}(x_1)$	$\mu_{\widetilde{A}}(x_2)$	$\mu_{\widetilde{A}}[\lambda x_1+(1-\lambda)x_2]$	$\min[(\mu_{\widetilde{A}}(x_1),\mu_{\widetilde{A}}(x_2)]$
0	0	0	0
0	1/4	(1-λ)/4	0
1/4	0	λ/4	0
1/4	1/4	¼	1/4
1/2	1/2	1/2	1/2
1/2	0	λ/2	0
0	1/2	(1- λ)/2	0
1/2	1	(2- λ)/2	1/2
1	1/2	(1+ λ)/2	1/2
1/2	1/4	(1+ λ)/4	1/4
1/4	1/2	(2- λ)/4	1/4
0	1	1-λ	0
1	1	1	1
1/4	1	(4- 3λ)/4	1/4
1	0	λ	0
1	1/4	(1+ 3λ)/4	1/4

From the above table, $\forall$ x_1, $x_2\in X$ and $\lambda\in[0,1]$, $\mu_{\widetilde{A}}[\lambda x_1+(1-\lambda)x_2]\ge\min[\mu_{\widetilde{A}}(x_1),\mu_{\widetilde{A}}(x_2)]$.

Hence the fuzzy set $\widetilde{A}$ corresponding to the classical family C_4 of finite type of Kac-Moody algebra is convex.

Now, consider the classical family D_4, whose Dynkin diagram is,



and the associated the GCM is, A = $\begin{pmatrix}2&-1&0&0\\-1&2&-1&-1\\0&-1&2&0\\0&-1&0&2\end{pmatrix}$

Here $\max|\langle\alpha_p,\alpha_q\rangle|=2$, for p, q = 1,2,3,4.

Let g(A) be the Kac – Moody Lie algebra associated with the GCM A. Let π be the set of simple roots of g(A). Let $\widetilde{A}=((\alpha_i,\alpha_j),\mu_{\widetilde{A}}(\alpha_i,\alpha_j))$ be the fuzzy set on $\pi\times\pi$ given by the equation (1).

The following lemmas describe some properties of this fuzzy set.

Lemma 8: Support of the fuzzy set $\widetilde{A}$ is $=\{(\alpha_1,\alpha_1),(\alpha_2,\alpha_2),(\alpha_3,\alpha_3),(\alpha_4,\alpha_4),(\alpha_1,\alpha_2),(\alpha_2,\alpha_1),(\alpha_2,\alpha_3),(\alpha_3,\alpha_2),(\alpha_2,\alpha_4),(\alpha_4,\alpha_2)\}$.

Proof: By the definition of support,

$Supp(\widetilde{A})=\{(\alpha_i,\alpha_j)/\mu_{\widetilde{A}}(\alpha_i,\alpha_j)>0\}$

$=\{(\alpha_1,\alpha_1),(\alpha_2,\alpha_2),(\alpha_3,\alpha_3),(\alpha_4,\alpha_4),(\alpha_1,\alpha_2),(\alpha_2,\alpha_1),(\alpha_2,\alpha_3),(\alpha_3,\alpha_2),(\alpha_2,\alpha_4),(\alpha_4,\alpha_2)\}$.

Lemma 9: Height of the fuzzy set $((\alpha_i,\alpha_j),\mu_{\widetilde{A}}(\alpha_i,\alpha_j))$ is 1.

Proof: Height of the fuzzy set is 1 because the maximum membership grade attained is 1.

Lemma 10: Core of the fuzzy set $\widetilde{A}$ is $\{(\alpha_1,\alpha_1),(\alpha_2,\alpha_2),(\alpha_3,\alpha_3),(\alpha_4,\alpha_4)\}$.

Proof: By the definition of core,

Core($\widetilde{A}$) = $\{(\alpha_i,\alpha_j)/\mu_{\widetilde{A}}(\alpha_i,\alpha_j)=1\}=\{(\alpha_1,\alpha_1),(\alpha_2,\alpha_2),(\alpha_3,\alpha_3),(\alpha_4,\alpha_4)\}$.

Lemma 11: The fuzzy set $\widetilde{A}$ is normal.

Proof: Since $Sup_{(\alpha_i,\alpha_j)}\mu_{\widetilde{A}}(\alpha_i,\alpha_j)=1$, the fuzzy set is normal.

Lemma 12: Let $\widetilde{A}$ be the fuzzy set defined on $\pi\times\pi$ for the finite Kac-Moody algebra D_4 given by equation (1) then $\widetilde{A}$ has the following properties:

(a) The cardinality $|\widetilde{A}|=7$.

(b) Relative cardinality $||\widetilde{A}||=7/16$

Proof: From equation (1), the fuzzy set $\widetilde{A}$ corresponding to the finite Kac-Moody algebra D_4 contains 1 element in X having membership grade 1, 5 elements in X having membership grade 1/2 and 4 elements in X having membership grade 1/4.

By the definition of cardinality,

(a) $|\widetilde{A}| = \sum_{x \in X} \mu_{\widetilde{A}}(x) = 7.$

(b) $\|\widetilde{A}\| = \frac{|\widetilde{A}|}{|X|} = \frac{7}{16}.$

Next we shall determine the α - level sets and Strong α - level sets for the specific values of α.

Computation of α – level sets:

Theorem 13: Let A be an indecomposable GCM of order 4 x 4, associated with the classical family D_4. Let $\widetilde{A}$ be the fuzzy set defined on $\pi \times \pi$ given by equation (1). Then the α - level sets and strong α - level sets for α = 1, 1/2, 1/3, ..., 1/k, ... are given by,

i. $A_1 = \{(\alpha_1,\alpha_1),(\alpha_2,\alpha_2),(\alpha_3,\alpha_3),(\alpha_4,\alpha_4)\}$, $|A_1| = 4$.

ii. $A_{1/2} = \{(\alpha_1,\alpha_1),(\alpha_2,\alpha_2),(\alpha_3,\alpha_3),(\alpha_4,\alpha_4),(\alpha_1,\alpha_2),(\alpha_2,\alpha_1),(\alpha_2,\alpha_3),$
$(\alpha_3,\alpha_2),(\alpha_2,\alpha_4),(\alpha_4,\alpha_2)\}, |A_{1/2}| = 10.$

iii. $A_{1/3} = A_{1/2}$, $A_1 \subset A_{1/2} = A_{1/3} = A_{1/4} = \ldots = A_{1/k} = \ldots,$
$|A_{1/k}| = 10,$ for $k = 2,3,\ldots$

iv. $A'_{1/2} = \{(\alpha_1,\alpha_1),(\alpha_2,\alpha_2),(\alpha_3,\alpha_3),(\alpha_4,\alpha_4)\},$
$|A'_{1/2}| = 4.$

v. $A'_{1/3} = \{(\alpha_1,\alpha_1),(\alpha_2,\alpha_2),(\alpha_3,\alpha_3),(\alpha_4,\alpha_4),(\alpha_1,\alpha_2),$
$(\alpha_2,\alpha_1),(\alpha_2,\alpha_3),(\alpha_3,\alpha_2),(\alpha_2,\alpha_4),(\alpha_4,\alpha_2)\},$
$|A'_{1/3}| = 10$

vi. $A'_{1/4} = A'_{1/3}$, and $A'_{1/2} \subset A'_{1/3} = A'_{1/4} = \ldots = A'_{1/k} = \ldots,$
$|A'_{1/k}| = 10$, for k = 3, 4,...

Proof: i. $A_1 = \{ (\alpha_i,\alpha_j) \in X / \mu_{\widetilde{A}}(\alpha_i,\alpha_j) \geq 1 \} =$
$\{(\alpha_1,\alpha_1),(\alpha_2,\alpha_2),(\alpha_3,\alpha_3),(\alpha_4,\alpha_4)\}.$

ii. $A_{1/2} = \{ (\alpha_i,\alpha_j) \in X / \mu_{\widetilde{A}}(\alpha_i,\alpha_j) \geq 1/2 \}$
$= A_1 \cup \{(\alpha_1,\alpha_2),(\alpha_2,\alpha_1),(\alpha_2,\alpha_3),(\alpha_3,\alpha_2),(\alpha_2,\alpha_4),(\alpha_4,\alpha_2)\}$
$|A_{1/2}| = 10.$

iii. $A_{1/3} = \{ (\alpha_i,\alpha_j) \in X / \mu_{\widetilde{A}}(\alpha_i,\alpha_j) \geq 1/3 \}$
$= A_1 \cup \{(\alpha_1,\alpha_2),(\alpha_2,\alpha_1),(\alpha_2,\alpha_3),(\alpha_3,\alpha_2),(\alpha_2,\alpha_4),(\alpha_4,\alpha_2)\} = A$

Also from the above result
$A_1 \subset A_{1/2} = A_{1/3} = A_{1/4} = \ldots = A_{1/k} = \ldots,$ $|A_{1/k}| = 10,$ $k = 2,3,\ldots$

iv. $A'_{1/2} = \{(\alpha_i,\alpha_j) \in X / \mu_{\widetilde{A}}(\alpha_i,\alpha_j) > 1/2\}$
$= \{(\alpha_1,\alpha_1),(\alpha_2,\alpha_2),(\alpha_3,\alpha_3),(\alpha_4,\alpha_4)\}, |A'_{1.2}| = 4.$

v. $A'_{1/3} = \{(\alpha_i,\alpha_j) \in X / \mu_{\widetilde{A}}(\alpha_i,\alpha_j) > 1/3\}$
$= \{(\alpha_1,\alpha_1),(\alpha_2,\alpha_2),(\alpha_3,\alpha_3),(\alpha_4,\alpha_4),(\alpha_1,\alpha_2),(\alpha_2,\alpha_1),$
$(\alpha_2,\alpha_3),(\alpha_3,\alpha_2),(\alpha_2,\alpha_4),(\alpha_4,\alpha_2)\}.$
$|A'_{1/3}| = 10.$

vi. $A'_{1/4} = \{(\alpha_i,\alpha_j) \in X / \mu_{\widetilde{A}}(\alpha_i,\alpha_j) > 1/4\} = A'_{1/3},$

Hence, $A'_{1/4} = A'_{1/3} = \ldots = A'_{1/k} = \ldots,$

From the above results,
$A'_{1/2} \subset A'_{1/3} = A'_{1/4} = \ldots = A'_{1/k} = \ldots$

Since $A'_{1/3} = A'_{1/k}$, $|A'_{1/k}| = 10$, for $k = 3,4,\ldots$

Lemma 14: A fuzzy set $\widetilde{A}$ corresponding to the finite type of classical family D_4, defined by (1) is convex.

Proof: Consider the finite type of classical family D_4. The table showing all possible membership grades for the elements of X and the conditions for checking convexity is listed below:

$\mu_{\widetilde{A}}(x_1)$	$\mu_{\widetilde{A}}(x_2)$	$\mu_{\widetilde{A}}[\lambda x_1 + (1-\lambda)x_2]$	$\min[(\mu_{\widetilde{A}}(x_1), \mu_{\widetilde{A}}(x_2)]$
0	0	0	0
1	1	1	1
1/2	0	λ/2	0
0	1/2	(1-λ)/2	0
1/2	1	(2 – λ)/2	1/2
1	1/2	(1+λ)/2	1/2
1/2	1/2	1/2	1/2
0	1	(1-λ)	0
1	0	λ	0

From the above table, ∀ $x_1, x_2 \in X$ and $\lambda \in [0, 1]$,
$\mu_{\widetilde{A}}[\lambda x_1 + (1-\lambda)x_2] \geq \min[\mu_{\widetilde{A}}(x_1), \mu_{\widetilde{A}}(x_2)].$

Hence the fuzzy set $\widetilde{A}$ corresponding to the classical family D_4 of finite type of Kac-Moody algebra is convex.

Theorem 15: Let $\widetilde{A}$ be the fuzzy set defined on $\pi \times \pi$, where π denotes the root basis for the finite Kac-Moody algebra given by equation (1). Then the α-cut decomposition for the fuzzy set $\widetilde{A}$ for the finite families are as follows:

(i) C_4: $1A_1 \cup 1/2A_{1/2} \cup 1/3A_{1/3} \cup 1/4A_{1/4}$.

(ii) D_4 : $1A_1 \cup 1/2A_{1/2}$.

Proof: (i) From the Theorem No [6] , for the finite family C_4,

We have, $A_1 \subset A_{1/2} = A_{1/3} \subset A_{1/4} = \ldots A_{1/k} = \ldots$

By the definition of α-cut decomposition,

The α-cut decomposition for the fuzzy set $\widetilde{A}$ for the finite family

C_4: $1A_1 \cup 1/2A_{1/2} \cup 1/3A_{1/3} \cup 1/4A_{1/4}$.

(ii) From the Theorem No [13] for the finite family D_4,

We have $A_1 \subset A_{1/2} = A_{1/3} = A_{1/4} = \ldots = A_{1/k} = \ldots$

The α-cut decomposition for the fuzzy set $\widetilde{A}$ for the finite family D_4 : $1A_1 \cup 1/2A_{1/2}$.

Conclusion: We can further compute the level cuts for various families of affine, hyperbolic and non hyperbolic of Kac-Moody algebras. Other structural properties on the fuzzy nature of these algebras can also be studied.

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