GRACEFUL LABELING ON THE COMBINATION OF SOME GRAPHS

S. SUDHA, V. KANNIGA

Abstract: Graceful labeling of a graph G with p vertices and q edges is an assignment of integer values to the vertices ranging from 0 to q so that the labels of edges is the absolute difference of the values assigned to the vertices ranging from 0 to q. Edges receive values from 1 to q. In this paper, we have shown that the arbitrary supersubdivision of wheel graphs is graceful. n-centipede graph is also proved to be graceful. A class of new graphs say $F_{2,n}$ is constructed by taking two fan graphs sharing their path in common and $F_{i,1,j}$ is obtained by considering two fan graphs sharing their core vertex in common is proved to be graceful. We also considered two star graphs S_m and S_n and joined some of the pendent vertices and proved the resulting graph is also graceful.

Keywords: Graceful labeling, Wheel graph, n-centipede graph, Fan graph, Double fan graph, Double star graph.

Introduction: Graceful labeling of the edges of the graph has got a tremendous results by considering various type of graphs. Gallian.J.A[1] has given a dynamic survey of graph labeling. Sethuraman.G and Selvaraju.P[2] have introduced the concept of supersubdivision of edges and proved that the arbitrary supersubdivision of paths and cycles is graceful. We have proved that the arbitrary supersubdivision of all the edges of wheel graphs by a complete bipartite graph $K_{2,n}$ for any n is graceful. We have also showed that the n-centipede graph is graceful. The fan graphs and the star graphs are obviously graceful. The double fan graph obtained by considering two fan graphs sharing their path in common is taken and proved to be graceful. We also considered the graph obtained by considering two fan graphs which share their core vertex as a common vertex and the graph is proved to be graceful. Further we consider the graph obtained by joining the pendent vertices of two star graphs and proved that this graph is also graceful.

2. Basic Definitions:

Definition 2.1 A graph $G = (V(G), E(G))$ with p vertices and q edges is said to admit *graceful labeling* if $f: V(G) \to \{0,1,2,\ldots,q\}$ such that distinct vertices receive distinct numbers and $\{|f(u) - f(v)| / uv \in E(G)\} = \{1,2,3,\ldots,q\}$.

Definition 2.2 A *wheel graph* W_n of order n, sometimes called as n-wheel, is a graph that consists of a cycle of order $n-1$. The edges of a wheel which include the hub are called spokes. It consists of n vertices and $2(n-1)$ edges. $W_n = K_1 + C_{n-1}$.

Definition 2.3 The *n-centipede* C_n is a tree with $2n$ vertices and $2n-1$ edges obtained by joining the bottoms of n copies of the path graph P_2 laid in a row with edges.

Definition 2.4 The join $K_1 \vee P_n$ of K_1 and P_n is called a *fan graph*, $F_{1,n}$. The vertex of K_1 is called the core. The edges incident with the core are called spokes.

Definition 2.5 The graph $\overline{K}_2 \vee P_n$, which is the join of complementary of K_2 and the path graph P_n is the *double fan graph* and is denoted by $F_{2,n}$. In other words, the double fan graph can be considered as the join of two similar fan graphs at the path.

Definition 2.6 *Double star graph* is a graph which is obtained by arbitrarily but consecutively joining the pendent vertices of two star graphs S_m and S_n.

3. Supersubdivision of wheel graph

Theorem 3.1 Arbitrary supersubdivision of wheel graphs are graceful.

Proof: Consider a wheel graph W_n which consists of n vertices and $2(n-1)$ edges. Supersubdivision of all edges of a wheel graph by the complete bipartite graph $K_{2,m}$ where m may vary in the resultant graph. This resultant graph can be labeled gracefully by defining a function f as follows: $f: V(G) \to \{0,1,2,\ldots,q\}$ where G is the resulting graph and q is the number of edges in the resulting graph.There is no general form of labeling this graph. This is illustrated by the following illustration.

Illustration 3.2 Consider a wheel graph, W_7. It consists of 7 vertices and 12 edges. Supersubdivision of all the edges of W_7 by $K_{2,2}, K_{2,3}, K_{2,4}, K_{2,5}, K_{2,6}$ is shown below. This graph is labeled gracefully as said in Theorem-3.1.

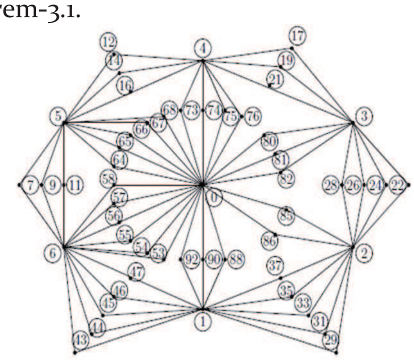

Figure 1 Supersubdivision of wheel graph W_7

4. n-Centipede graph

Theorem4.1 n-centipede(C_n)graph has graceful labeling for any n.

Proof: Let v_i , $i = 1,2,3,\dots,2n$ be the vertices of the n-centipede graph where the pendent vertices are marked as $v_1,\ v_2,\ v_3,\dots,v_n$ from left to right respectively and the other vertices be marked as $v_{n+1},\ v_{n+2},\ v_{n+3},\dots,\ v_{2n}$ from right to left respectively. Define the function f as follows:

$f: V(G) \to \{0,1,2,\dots,q\}$

$for\ i \le n,$

$$f(v_i) = \begin{cases} i-1, & if\ i\ is\ odd \\ q-i+1, & if\ i\ is\ even \end{cases}$$

$for\ i > n,$

$$f(v_i) = \begin{cases} q-i+1, & if\ i\ is\ odd \\ i-1, & if\ i\ is\ even \end{cases}$$

where q is the number of edges and i is a positive integer. By using the above definition the graceful labeling can be done for n-centipede graph.

Illustration 4.2 Consider the 5-centipede graph(C_5) and label it as defined in the above Theorem-4.1. This is illustrated in figure-2.

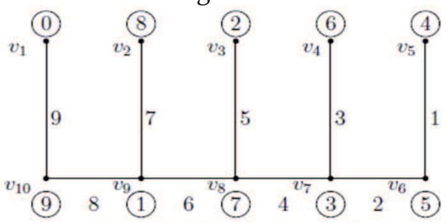

Figure 2 5-centipede graph

Illustration 4.3 Consider the 6-centipede graph(C_6) and label it as defined in the above Theorem-4.1. This is illustrated in figure-3.

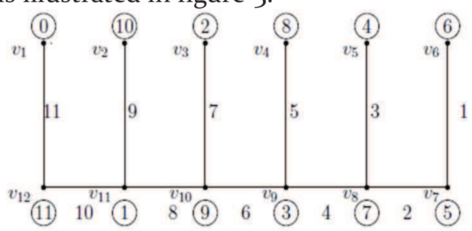

Figure 3 6-centipede graph

5. Double fan graph

Theorem 5.1 Double fan graph $F_{2,n}$ is a graceful graph.

Proof: The double fan graph $F_{2,n}$ consists of $n+2$ vertices and $3n-1$ edges. The vertices of $\overline{K}_2$ are denoted as v_0 and v_{n+1}. The vertices of P_n are denoted as $v_1,\ v_2, v_3,\dots,\ v_n$. Define the function f as follows:

$f: V(G) \to \{0,1,2,\dots,q\}$

where q is the number of edges in the graph.

$f(v_0) = 0$

$f(v_{n+1}) = n$

$for\ 1 \le\ i\ \le n,$

$$f(v_i) = \begin{cases} q-\left(\frac{i-1}{2}\right), & if\ i\ is\ odd \\ q-n+\frac{i}{2}, & if\ i\ is\ even \end{cases}$$

By using this definition of f, the double fan graph $F_{2,n}$ is labeled gracefully.

Illustration 5.2 (n odd)

The double fan graph $F_{2,5}$ consists of 7 vertices and 14 edges. This graph is labeled gracefully by using Theorem-5.1 and is shown in figure-4.

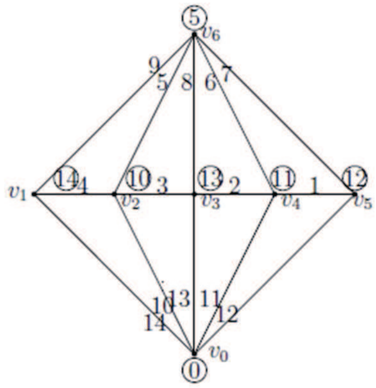

Figure 4 Double fan graph $F_{2,5}$

Illustration 5.3 (n even)

The double fan graph $F_{2,6}$ consists of 8 vertices and 17 edges as shown in figure-5. This graph is labeled gracefully by using Theorem-5.1.

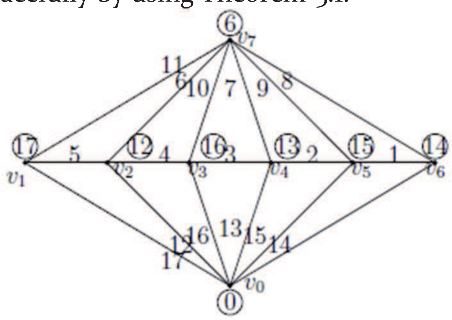

Figure 5 Double fan graph $F_{2,6}$

Theorem 5.4 Two fan graphs sharing their core in common is a graceful graph.

Proof: Consider two fan graphs $F_{1,i}$ and $F_{1,j}$. A new graph $F_{i,1,j}$ is obtained when two fan graphs $F_{1,i}$ and $F_{1,j}$ share their core in common. This resultant graph $F_{i,1,j}$ consists of $(i+j+1)$ vertices and $2(i+j-1)$ edges, where $i = n-1, n+1, n+2$ and $j = n$.

Case 1: $F_{n-1,1,n}$

We consider two fan graphs $F_{1,n-1}$ and $F_{1,n}$. If the vertices of $F_{1,n}$ are denoted as $v_1,\ v_2,\ v_3,\dots,\ v_n$ from left to right respectively and the core vertex denoted as v_0 and the vertices of $F_{1,n-1}$ are denoted as $v_{n+1},\ v_{n+2},\ v_{n+3},\dots,\ v_{2n-1}$ from right to left respectively, we define the function f as follows: $f: V(G) \to \{0,1,2,\dots,q\}$

$f(v_0) = 0$

The vertices $v_1,\ v_2,\ v_3,\dots,\ v_n$ are labeled as

$$f(v_i) = \begin{cases} q-\left(\frac{i-1}{2}\right), & if\ i\ is\ odd \\ (p-2)+\frac{i}{2}, & if\ i\ is\ even \end{cases}$$

where, p is the number of vertices and q is the number of edges in the resultant graph. The rest of the vertices $v_{n+1}, v_{n+2}, v_{n+3}, \dots, v_{2n-1}$ are labeled in such a way that the labeling is graceful.

Case 2: $F_{n+1,1,n}$

We consider two fan graphs $F_{1,n+1}$ and $F_{1,n}$. If the vertices of $F_{1,n}$ are denoted as $v_1, v_2, v_3, \dots, v_n$ from left to right respectively and the core vertex denoted as v_0 and the vertices of $F_{1,n+1}$ are denoted as $v_{n+1}, v_{n+2}, v_{n+3}, \dots, v_{2n+1}$ from right to left respectively, we define the function f as follows: $f: V(G) \to \{0,1,2,\dots,q\}$

where q is the number of edges in the graph.

$f(v_o) = 0$

$f(v_{2n+1}) = p - 2$

If n is odd then the vertices $v_1, v_3, v_5, \dots, v_n$ are labeled as $q, q-1, q-2, \dots, q-\left(\frac{n-1}{2}\right)$. The vertices $v_2, v_4, v_6, \dots, v_{n-1}$ are labeled as $p-1, p, p+1, \dots, (p-2)+\left(\frac{n-1}{2}\right)$. The vertices $v_{n+1}, v_{n+3}, v_{n+5}, \dots, v_{2n}$ are labeled as $q-\left(\frac{n-1}{2}\right)-1, q-\left(\frac{n-1}{2}\right)-2, q-\left(\frac{n-1}{2}\right)-3, \dots$. The vertices $v_{n+2}, v_{n+4}, v_{n+6}, \dots, v_{2n-1}$ are labeled as $(p-2)+\left(\frac{n-1}{2}\right)+1, (p-2)+\left(\frac{n-1}{2}\right)+2, (p-2)+\left(\frac{n-1}{2}\right)+3, \dots$. If n is even then the vertices $v_1, v_3, v_5, \dots, v_{n-1}$ are labeled as $q, q-1, q-2, \dots, q-\left(\frac{n-2}{2}\right)$. The vertices $v_2, v_4, v_6, \dots, v_n$ are labeled as $p-1, p, p+1, \dots, (p-2)+\left(\frac{n}{2}\right)$. The vertices $v_{n+1}, v_{n+3}, v_{n+5}, \dots, v_{2n-1}$ are labeled as $q-\left(\frac{n-2}{2}\right)-1, q-\left(\frac{n-2}{2}\right)-2, q-\left(\frac{n-2}{2}\right)-3, \dots$. The vertices $v_{n+2}, v_{n+4}, v_{n+6}, \dots, v_{2n}$ are labeled as $(p-2)+\left(\frac{n}{2}\right)+1, (p-2)+\left(\frac{n}{2}\right)+2, (p-2)+\left(\frac{n}{2}\right)+3, \dots$ where p is the number of vertices and q is the number of edges in the resultant graph.

Case 3: $F_{n+2,1,n}$

We consider two fan graphs $F_{1,n+2}$ and $F_{1,n}$. If the vertices of $F_{1,n}$ are denoted as $v_1, v_2, v_3, \dots, v_n$ from left to right respectively and the core vertex denoted as v_0 and the vertices of $F_{1,n+2}$ denoted as $v_{n+1}, v_{n+2}, v_{n+3}, \dots, v_{2n+2}$ from right to left respectively, we define the function f as follows: $f: V(G) \to \{0,1,2,\dots,q\}$.

$f(v_o) = 0$

$f(v_{2n+2}) = p - 2$

The vertices $v_1, v_3, v_5, \dots, v_{n-1}$ are labeled as $q, q-1, q-2, \dots, q-\left(\frac{n-2}{2}\right)$. The vertices $v_2, v_4, v_6, \dots, v_n$ are labeled as $p-1, p, p+1, \dots, (p-2)+\left(\frac{n}{2}\right)$. The vertices $v_{n+1}, v_{n+3}, v_{n+5}, \dots, v_{2n+1}$ are labeled as $(p-2)+\left(\frac{n}{2}\right)+1, (p-2)+\left(\frac{n}{2}\right)+2, (p-2)+\left(\frac{n}{2}\right)+3, \dots$. The vertices $v_{n+2}, v_{n+4}, v_{n+6}, \dots, v_{2n}$ are labeled as as $q-\left(\frac{n-2}{2}\right)-1, q-\left(\frac{n-2}{2}\right)-2, q-\left(\frac{n-2}{2}\right)-3, \dots$ where p is the number of vertices and q is the number of edges in the resultant graph. This result holds only for even n.

Illustration 5.5 Consider $F_{6,1,7}$ for odd $n = 7$. We label this graph gracefully by using case-1 of Theorem-5.4.

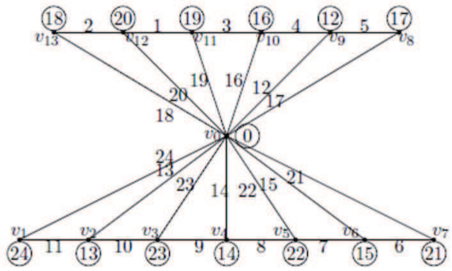

Figure 6 $F_{6,1,7}$

Illustration 5.6 Consider $F_{9,1,8}$. Here $n = 8$. We label this graph gracefully by using case-2 of Theorem-5.4.

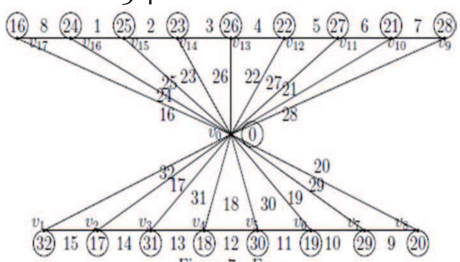

Figure 7 $F_{9,1,8}$

Illustration 5.7 Consider $F_{8,1,6}$. We label this graph gracefully by using case-3 of Theorem-5.4.

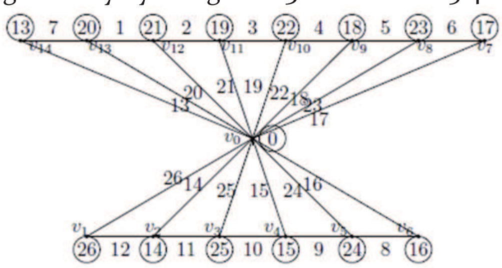

Figure 8 $F_{8,1,6}$

6. Double star graph

Theorem 6.1 Double star graph is graceful.

Proof: Consider two star graphs S_m and S_n. If k pendent vertices of S_m and S_n are merged as per the definition-2.6, we get the double star graph which consists of $(m+n)-k+2$ vertices and $m+n$ edges. We label the new graph gracefully by defining the function f as follows: $f: V(G) \to \{0,1,2,\dots,q\}$ where, q is the number of edges in the resultant graph. The center vertex of S_m say, v_0' is labeled as k and the center vertex of S_n say, v_0 is labeled as 0. The other n vertices say $v_1, v_2, v_3, \dots, v_n$ of S_n are labeled as $(m+n), (m+n-1), (m+n-2), \dots, (m+1)$. The $m-k$ vertices of S_m say, $v_1', v_2', v_3', \dots, v_{m-k}'$ are labeled as $m, m-1, m-2, \ldots, (k+1)$. With this type of labeling, we label the resultant graph gracefully.

Illustration 6.2 $(m < n)$

Consider two star graphs S_4 and S_8. If three pendent vertices of these star graphs are merged together then the resulting double star graph consists of 11 vertices and 12 edges. We label this graph gracefully by using Theorem-6.1 and is shown in figure-9.

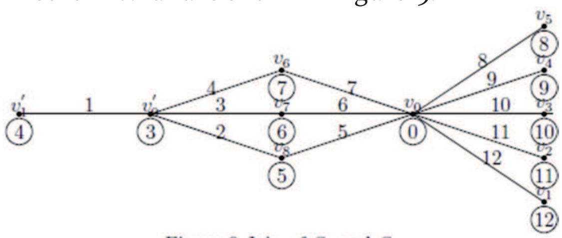

Figure 9 Join of S_4 and S_8

Conclusion

We have shown that arbitrary supersubdivision of a wheel graph, n-centipede graph, two fan graphs sharing either their path or their core in common are graceful. We have also shown that the join of two star graphs S_m and S_n at their pendent vertices are graceful.

References

1. Gallian, J.A.(2011). A Dynamic Survey of Graph Labeling. *The Electronic Journal of Combinatorics, 18, #DS6.*
2. Sethuraman, G. and Selvaraju, P., Gracefulness of arbitrary supersubdivision of graphs, *Indian J.Pure appl.Math.,*32(7):1059 - 1064, July 2001.
3. Rosa, A.(1967). On Certain Valuations of the Vertices of a Graph, In: Theory of Graphs, *(International Symposium, Rome, July 1966). Gordon and Breach, N.Y. and Dunod Paris*, 349 - 355.
4. Golomb, S. W.(1972). How to number a graph. In: *Graph Theory and Computing(R. C. Read. Ed.)* Academic Press, New York, 23 - 37.

***

Professor of Mathematics, Ramanujan Institute for Advanced Study in Mathematics
University of Madras, Chennai-600 005.
E-mail: ssudha50@sify.com
Research Scholar,Ramanujan Institute for Advanced Study in Mathematics
University of Madras, Chennai-600 005.
E-mail: kan15gemini@gmail.com