SUPERPOSITION OF STARS ON CYCLES AND n-CENTIPEDES ARE GRACEFUL

S. SUDHA, V. KANNIGA

Abstract: In this paper we prove the new class of graph denoted by $C_m \mathbin{\$} S_n$ which is obtained by identifying any vertex ofC_m, $m \geq 3$ with any vertex of S_n is graceful. In addition, we have proved that the superposition of star on a n-centipede graph is graceful.

Keywords: Graceful labeling, Superposition, n-centipede graph.

Introduction: Swaminathan.V and Sekar.C[1] have proved that the superposition of stars on cycles is graceful for $n \equiv 1(mod\ 4)$ and $n \equiv 2(mod\ 4)$. If S_n is a star with n edges and $n+1$ vertices and if C_m is a cycle with m edges and m vertices, then the new class of graph G is obtained by identifying any vertex of $C_m$with any vertex of S_n. The resulting graph G is proved to be graceful. We have also considered the superposition of stars on a n-centipede graph and proved it to be graceful.

Definition 1.1 A graph $G = (V(G), E(G))$ with p vertices and q edges is said to admit *graceful labeling* if $f: V(G) \to \{0,1,2,\dots,q\}$ such that distinct vertices receive distinct numbers and $\{|f(u) - f(v)|/ uv \in E(G)\} = \{1,2,3,\dots,q\}$.

Definition 1.2 The *superposition* of two graphs G_1 and G_2 denoted by $G = G_1 \mathbin{\$} G_2$ and is defined to be the graph with vertex set $V = V_1 \cup V_2$ and the edge set containing all the edges of G_1 and G_2 by identifying one of the vertex of $G_1$with one of the vertex of G_2 in G.

Definition 1.3 The *n-centipede* (C_n) is a tree with $2n$ vertices and $2n-1$ edges obtained by joining the bottoms of n copies of the path graph P_2 laid in a row with edges.

2. Superposition of a star on a cycle

Theorem 2.1 For $n \equiv m(mod\ j)$, where $m = 0,1,2,\dots,(j-1)$ the graph G_n is graceful. Here n takes values from $j+m, 2j+m, 3j+m, 4j+m, \dots$

Proof: There are two cases - one if n is an odd integer and another if n is an even integer.

Case 1: n be an odd integer

The new class of graph G_n is denoted by $C_n \mathbin{\$} S_{\frac{n-1}{2}}$ has $n + (\frac{n-1}{2})$ vertices and $n + (\frac{n-1}{2})$ edges. Let u_i, $i = 1,2,3,\dots,n$ be the vertices of C_n and v_i, $i = 0,1,2,\dots,\frac{n-1}{2}$ be the vertices of $S_{\frac{n-1}{2}}$ with v_0 as the center of the star. The vertices of the cycle and the star are identified if $u_n = v_{\frac{n-1}{2}}$, we define the function f on the vertices of $G_n$as follows:

$$f(u_i) = \begin{cases} \dfrac{i-1}{2}, & i = 1,3,5,\dots,n-2 \\ n + \left(\dfrac{n-1}{2}\right) - \dfrac{i}{2}, & i = 2,4,6,\dots,n-1 \\ n + \left(\dfrac{n-1}{2}\right), & i = n \end{cases}$$

with this definition of f the vertices of the graph $S_{\frac{n-1}{2}}$ are labeled as follows:

$$f(v_i) = \begin{cases} n-1, & i = 0 \\ n-1-i, & i = 1,2,3,\dots,\left(\dfrac{n-1}{2} - 1\right) \\ n + \left(\dfrac{n-1}{2}\right), & i = \dfrac{n-1}{2} \end{cases}$$

Now identify any one of the vertex of C_n with any vertex of $S_{\frac{n-1}{2}}$ to obtain the graph $C_n \mathbin{\$} S_{\frac{n-1}{2}}$. It is obvious that f is a one-to-one map from the vertex set of $G_n$into the set $\{0,1,2,\dots,q\}$ where $q = n + \frac{n-1}{2}$.

Case 2: n be an even integer

The new class of graph G_n is denoted by $C_n \mathbin{\$} S_{\frac{n-2}{2}}$ has $\frac{n}{2} + (n-1)$ vertices and $\frac{n}{2} + (n-1)$ edges. Let u_i, $i = 1,2,3,\dots,n$ be the vertices of C_n and v_i, $i = 0,1,2,\dots,\frac{n-2}{2}$ be the vertices of $S_{\frac{n-2}{2}}$ with v_0 as the center of the star. The vertices of the cycle and the star are identified if $u_n = v_{\frac{n-2}{2}}$, we define the function f_1 on the vertices of G_n as follows:

$$f_1(u_i) = \begin{cases} \dfrac{i-1}{2}, & i = 1,3,5,\dots,n-3 \\ \left(\dfrac{3n-i}{2}\right) - 1, & i = 2,4,6,\dots,n-2 \\ i, & i = n-1 \\ \left(\dfrac{3n}{2}\right) - 1, & i = n \end{cases}$$

with this definition of f_1 the vertices of the graph $S_{\frac{n-2}{2}}$ are labeled as follows:

$$f_1(v_i) = \begin{cases} n-2, & i = 0 \\ n-2-(i+1), & i = 1,2,3,\dots,\left(\dfrac{n-2}{2} - 1\right) \\ \left(\dfrac{3n}{2}\right) - 1, & i = \dfrac{n-2}{2} \end{cases}$$

Now identify any one of the vertex of C_n with any vertex of $S_{\frac{n-2}{2}}$ to obtain the graph $C_n \mathbin{\$} S_{\frac{n-2}{2}}$. Again f_1 is a one-to-one map from the vertex set of G_n into the set $\{0, 1, 2, \dots, q\}$ where $q = \frac{n}{2} + (n-1)$. We illustrate the above theorem with the following examples.

Illustration 2.2 Consider the case when $j = 4$. The graceful numbering is of the form $n \equiv m(mod\ 4)$.

Since m takes the values 0, 1, 2 and 3 we consider the following:

(i) $n \equiv 0(mod\ 4)$, where n may take any one of the values $4, 8, 12, 16, \dots$

(ii) $n \equiv 1(mod\ 4)$, where n may take any one of the values $5, 9, 13, 17, \dots$

(iii) $n \equiv 2(mod\ 4)$, where n may take any one of the values $6, 10, 14, 18, \dots$

(iv) $n \equiv 3(mod\ 4)$, where n may take any one of the values $7, 11, 15, 19, \dots$

(i) Let us consider $n \equiv 0(mod\ 4)$ then $n = 4, 8, 12, 16, \dots$ which are multiples of 4 in this case.

Let n be 12 even, the cycle graph is C_{12} and star graph is S_5 according to the definition. The new class of graph G_{12} is C_{12} ṡ S_5 and is drawn in figure-1, where the vertices are numbered by using Theorem-2.1.

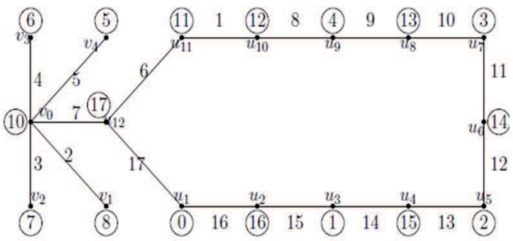

Figure 1. $G_{12} = C_{12}$ ṡ S_5

(ii) Let us consider $n \equiv 1(mod\ 4)$ then $n = 5, 9, 13, 17, \dots$

Let n be 5 odd, the cycle graph is C_5 and star graph is S_2 according to the definition. The new class of graph G_5 is C_5 ṡ S_2 and is drawn in figure-2, where the vertices are numbered by using Theorem-2.1.

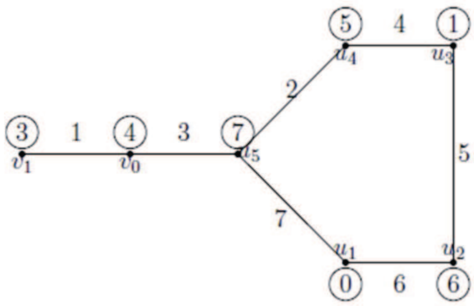

Figure 2. $G_5 = C_5$ ṡ S_2

(iii) Let us consider $n \equiv 2(mod\ 4)$ then $n = 6, 10, 14, 18, \dots$

Let n be 14 even, the cycle graph is C_{14} and star graph is S_6 according to the definition. The new class of graph G_{14} is C_{14} ṡ S_6 and is drawn in figure-3, where the vertices are numbered by using Theorem-2.1.

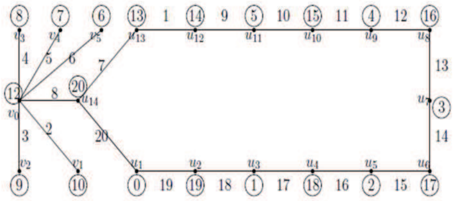

Figure 3. $G_{14} = C_{14}$ ṡ S_6

(iv) Let us consider $n \equiv 3(mod\ 4)$ then $n = 7, 11, 15, 19, \dots$ Let n be 11 odd, the cycle graph is C_{11} and star graph is S_5 according to the definition. The new class of graph G_{11} is C_{11} ṡ S_5 and is drawn in figure-4, where the vertices are numbered by using Theorem-2.1.

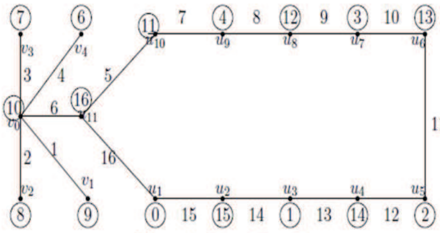

Figure 4. $G_{11} = C_{11}$ ṡ S_5

3. Superposition of a star on a n-centipede graph

Theorem 3.1 The superposition of a star graph(S_k) on a n-centipede(C_n) graph is graceful.

Proof: We know that the n-centipede graph and the star graph is always graceful. Let $v_1, v_2, v_3, \dots, v_{2n}$ be the vertices of the n-centipede graph. The pendent vertices of the n-centipede graph are denoted as $v_1, v_2, v_3, \dots, v_n$ from left to right respectively and the other vertices of the n-centipede graph are denoted as $v_{n+1}, v_{n+2}, v_{n+3}, \dots, v_{2n}$ from right to left respectively. The vertices of the star graph are denoted as $v_{2n+1}, v_{2n+2}, v_{2n+3}, \dots, v_{2n+k}$ with v_0 as the center vertex. We now consider a new class of graph by identifying the initial vertex of n-centipede graph and the central vertex of the star graph that is if $v_1 = v_0$. The resultant graph consists of $2n + k$ vertices and $2n + k - 1$ edges. We define the function f as follows: $f: V(G) \to \{0,1,2,\dots,q\}$

$f(v_1) = f(v_0) = 0$

$for\ 2 \le i \le n,$

$$f(v_i) = \begin{cases} i-1, & if\ i\ is\ odd \\ q-(i+k)+1, & if\ i\ is\ even \end{cases}$$

$for\ n+1 \le i \le 2n,$

$$f(v_i) = \begin{cases} q-(i+k)+1, & if\ i\ is\ odd \\ i-1, & if\ i\ is\ even \end{cases}$$

$for\ 2n+1 \le i \le 2n+k,$

$f(v_i) = i - 1, for\ both\ odd\ and\ even\ i.$

here, q is the number of edges in the resultant graph. By using this definition of f, graceful labeling can be given to the newly obtained graph.

Illustration 3.2 Consider the 3-centipede graph (C_3) and it is superpositioned by a star graph S_4 and the labeling is done as shown in figure-5 by using

Theorem-3.1.

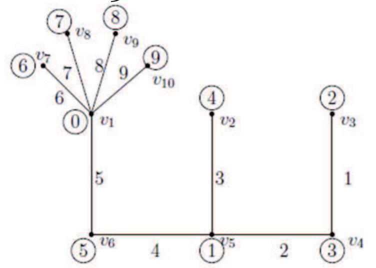

Figure 5. Superposition of 3-centipede by a star graph S_4

Illustration 3.3 Consider the 6-centipede graph(C_6) and it is superpositioned by a star graph S_7 and the labeling is done as shown in figure-6 by using Theorem-3.1.

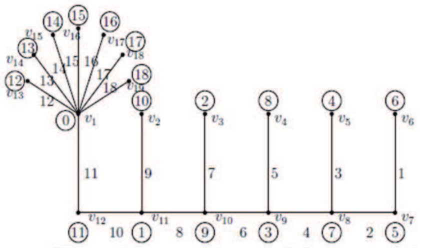

Figure 6. Superposition of 6-centipede by a star graph S_7

Trees 3.4 The graph $C_n - \{v_n\}$ which is obtained from the n-centipede graph $C_n$by removing a vertex at the end, that is the peripheral vertex can also be labeled gracefully. Let $v_1, v_2, v_3, \ldots, v_{2n-1}$ be the vertices of the resulting graph where the pendent vertices are denoted as $v_1, v_2, v_3, \ldots, v_n$ from left to right respectively and the other vertices are denoted as $v_{n+1}, v_{n+2}, v_{n+3}, \ldots, v_{2n-1}$ from right to left respectively. This graph consists of $2n - 1$ vertices and $2n - 2$ edges. This graph is labeled gracefully by using the following definition of f as follows: $f: V(G) \to \{0,1,2,\ldots,q\}$
for $1 \leq i \leq 2n - 1$,

$$f(v_i) = \begin{cases} i - 1, & if\ i\ is\ odd \\ q - i + 1, & if\ i\ is\ even \end{cases}$$

where q is the number of edges in the resulting graph.

Illustration 3.5 Consider the 7-centipede graph(C_7) without a peripheral vertex v_7 and its labeling is done as shown in figure-7 by using the above result-3.4.

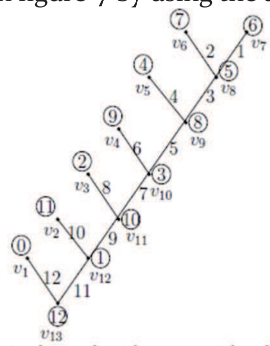

Figure 7. 7-centipede graph without a peripheral vertex v_7

Illustration 3.6 Consider the 8-centipede graph(C_8) without a peripheral vertex v_8 and its labeling is done as shown in figure-8 by using the above result-3.4.

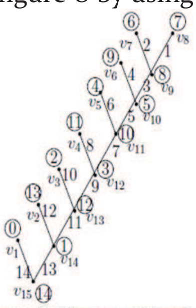

Figure 8. 8-centipede graph without a peripheral vertex v_8

4. Conclusion:

We proved here that the superposition of stars on cycles and superposition of stars on n-centipede graphs is graceful.

References:

1. Swaminathan, V. and Sekar, C., A New Class of Graceful Graphs, *Graph Theory and its Applications(2004)* (84 - 88).
2. Gallian, J.A.(2011). A Dynamic Survey of Graph Labeling. *The Electronic Journal of Combinatorics, 18,* #DS6.
3. Golomb, S. W.(1972). How to number a graph. In: *Graph Theory and Computing*(R. C. Read. Ed.) Academic Press, New York, 23 - 37.
4. Rosa, A.(1967). On Certain Valuations of the Vertices of a Graph, *In: Theory of Graphs, (International Symposium, Rome, July 1966).* Gordon and Breach, N.Y. and Dunod Paris, 349 - 355.

***

Professor of Mathematics, Ramanujan Institute for Advanced Study in Mathematics
University of Madras, Chennai-600 005.
E-mail: ssudha50@sify.com
Research Scholar,Ramanujan Institute for Advanced Study in Mathematics
University of Madras, Chennai-600 005.
E-mail: kan15gemini@gmail.com