THE *p*– METRIC SPACE OF Γ^2 DEFINED BY MUSIELAK

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Abstract: In the present paper we introduce the *p*– metric space of Γ^2 defined by a Musielak modulus function. We study some topological properties and prove some inclusion relations between these spaces. Lindenstrauss and Tzafriri [5] used the idea of Orlicz function to define the sequence space l_M which is called an Orlicz sequence space. Another generalization of Orlicz sequence spaces is due to Woo [31].

Keywords :. analytic sequence, modulus function, double sequences, χ^2 space, difference sequence space, Musielak - modulus function, *p*– metric space, duals.

Introduction : Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively. We write w^2 for the set of all complex sequences (x_{mn}), where $m, n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication. Some initial works on double sequence spaces is found in Bromwich [2]. Later on, they were investigated by Hardy [3], Moricz [7], Moricz and Rhoades [8], Basarir and Solankan [1], Tripathy [11], Turkmenoglu [12], and many others.We procure the following sets of double sequences:

$$M_u(t) := \left\{ (x_{mn}) \in w^2 : \sup_{m,n\in N} |x_{mn}|^{t_{mn}} < \infty \right\},$$

$$C_p(t) = \left\{ (x_{mn}) \in w^2 : p-\lim_{m,n\to\infty} |x_{mn} - l|^{t_{mn}} = 1 \; for\, some\, l \in C \right\},$$

$$C_{0p}(t) := \left\{ (x_{mn}) \in w^2 : p-\lim_{m,n\to\infty} |x_{mn}|^{t_{mn}} = 1 \right\},$$

$$L_u(t) := \left\{ (x_{mn}) \in w^2 : \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |x_{mn}|^{t_{mn}} < \infty \right\},$$

$$C_{bp}(t) := C_p(t) \cap M_u(t) \text{ and } C_{0bp}(t) = C_{0p}(t) \cap M_u(t);$$

where $t = (t_{mn})$ is the sequence of strictly positive reals t_{mn} for all $m, n \in \mathbb{N}$and $p-\lim_{m,n\to\infty}$ denotes the limit in the Pringsheim's sense. In the case $t_{mn} = 1$ for all $m, n \in \mathbb{N}$; $M_u(t)$, $C_p(t)$, $C_{0p}(t)$, $L_u(t)$, $C_{bp}(t)$ and $C_{0bp}(t)$ reduce to the sets M_u, C_p, C_{0p}, L_u, C_{bp} and C_{0bp}, respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. Gokhan and Colak [14,15] have proved that $M_u(t)$ and $C_p(t)$, $C_{bp}(t)$ are complete paranormed spaces of double sequences and gave the α–, β–, γ– duals of the spaces $M_u(t)$ and $C_{bp}(t)$. Quite recently, in her PhD thesis. Zelter [16] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [17] and Tripathy [11] have independently introduced the statistical convergence and Cauchy for double sequences and given the relation between statistical convergent and strongly Cesaro summable double sequences. Altay and Basar [20] have defined the spaces BS, $BS(t)$, CS_p, CS_{bp}, CS_r and BV of double sequences consisting of all double series whose sequence of partial sums are in the spaces M_u, $M_u(t)$, C_p, C_{bp}, C_r and L_u, respectively, and also examined some properties of those sequence spaces and determined the α– duals of the spaces BS, BV, CS_{bp} and the $\beta(\nu)$ – duals of the spaces CS_{bp} and CS_r of double series. Basar and Sever [21] have introduced the Banach space L_q of double sequences corresponding to the well-known space l_q of single sequences and examined some properties of the space L_q. Quite recently Subramanian and Misra [22] have studied the space $\chi^2_M(p,q,u)$ of double sequences and gave some inclusion relations.

The class of sequences which are strongly Cesaro summable with respect to a modulus was introduced by Maddox [6] as an extension of the definition of strongly Cesaro summable sequences. Connor [23] further extended this definition to a definition of strong A– summability with respect to a modulus where $A = (a_{n,k})$ is a nonnegative regular matrix and established some connections between strong A– summability, strong A– summability with respect to a modulus, and A– statistical convergence. In [24] the notion of convergence of double sequences was presented by A. Pringsheim. Also, in [25]-[26] and [27] the four dimensional matrix transformation $(Ax)_{k,l} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{kl}^{mn} x_{mn}$ was studied extensively by Robison and Hamilton.

We need the following inequality in the sequel of the paper. For $a, b \geq 0$ and $0 < p < 1$, we have

(1.1) $$(a+b)^p \leq a^p + b^p$$

The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is called convergent if and only if the double sequence (s_{mn}) is convergent, where $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij}$ $(m, n \in N)$.

A sequence $x = (x_{mn})$ is said to be double analytic if $\sup_{mn} |x_{mn}|^{1/m+n} < \infty$. The vector space of all double analytic sequences will be denoted by Λ^2. A sequence $x = (x_{mn})$ is called double gai sequence if $\left((m+n)!|x_{mn}|\right)^{1/m+n} \to 0$ as, $m, n \to \infty$. The double gai sequences will be denoted by χ^2. Let ϕ = {*all finite sequences*}.

Consider a double sequence $x = (x_{ij})$. The $(m, n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij}\Im_{ij}$ for all $m, n \in$N; where $\Im_{ij}$ denotes the double sequence whose only non zero term is a $\frac{1}{(i+j)!}$ in the $(i, j)^{th}$ place for each $i, j \in$ N.

An FK-space (or a metric space) X is said to have AK property if $(\Im_{mn})$ is a Schauder basis for X. Or equivalently $x^{[m,n]} \to x$.

An FDK-space is a double sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mapping $x = (x_k) \to (x_{mn})$ $(m, n \in$ N) are also continuous.

Let M and Φ are mutually complementary modulus functions. Then, we have:

(i) For all $u, y \geq 0$,

(1.2) $$uy \leq M(u) + \Phi(y), (Young's\, inequality)[See [13]]$$

(ii) For all $u \geq 0$,

(1.3) $$u\eta(u) = M(u) + \Phi(\eta(u))$$

(iii) For all $u \geq 0$, and $0 < \lambda < 1$.

(1.4) $$M(\lambda u) \leq \lambda M(u)$$

Lindenstrauss and Tzafriri [5] used the idea of Orlicz function to construct Orlicz sequence space

$$l_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\},$$

The space l_M with the norm $$\|x\| = \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\},$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p (1 \leq p < \infty)$, the spaces l_M coincide with the classical sequence space l_p.

A sequence $f = (f_{mn})$ of modulus function is called a Musielak-modulus function. A sequence $g = (g_{mn})$ defined by $$g_{mn}(v) = \sup\{|v|u - (f_{mn})(u) : u \geq 0\}, m, n = 1, 2, ...$$

is called the complementary function of a Musielak-modulus function f. For a given Musielak modulus function f, the Musielak-modulus sequence space t_f and its sub-space h_f are defined as follows

$$t_f = \left\{ x \in w^2 : I_f\left(|x_{mn}|\right)^{1/m+n} \to 0 \text{ as } m, n \to \infty \right\},$$

$$h_f = \left\{ x \in w^2 : I_f\left(|x_{mn}|\right)^{\frac{1}{m+n}} \to 0 \text{ as } m, n \to \infty \right\},$$

where I_f is a convex modular defined by

$$I_f(x) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_{mn}\left(|x_{mn}|\right)^{1/m+n}, x = (x_{mn}) \in t_f.$$

We consider t_f equipped with the Luxemburg metric

$$d\left(x,y\right)=\sup_{mn}\left\{\inf\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}f_{mn}\left(\frac{\left|x_{mn}\right|^{1/m+n}}{mn}\right)\right)\leq 1\right\}$$

If X is a sequence space, we give the following definitions:

(i) X' = the continuous dual of X;

(ii) $X^{\alpha}=\left\{a=\left(a_{mn}\right):\sum_{m,n=1}^{\infty}\left|a_{mn}x_{mn}\right|<\infty,\ for\ each\ x\in X\right\}$;

(iii) $X^{\beta}=\left\{a=\left(a_{mn}\right):\sum_{m,n=1}^{\infty}a_{mn}x_{mn}\ is\ convegent,\ for\ each\ x\in X\right\}$;

(iv) $X^{\gamma}=\left\{a=\left(a_{mn}\right):\sup_{mn}\geq 1\left|\sum_{m,n=1}^{M,N}a_{mn}x_{mn}\right|<\infty,\ for\ each\ x\in X\right\}$;

(v) let X be an FK-space $\supset \phi$; then $X^{f}=\left\{f\left(\Im_{mn}\right):f\in X'\right\}$;

(vi) $X^{\delta}=\left\{a=\left(a_{mn}\right):\sup_{mn}\left|a_{mn}x_{mn}\right|^{1/m+n}<\infty,\ for\ each\ x\in X\right\}$;

$X^{\alpha}.X^{\beta}$, X^{γ} are called α – (or Kothe – Toeplitz) dual of X, β – (or generalized – Kothe – Toeplitz) dual of X, γ – dual of X, δ – dual of X respectively. X^{α} is defined by Gupta and Kamptan [13]. It is clear that $X^{\alpha}\subset X^{\beta}$ and $X^{\alpha}\subset X^{\gamma}$, but $X^{\beta}\subset X^{\gamma}$ does not hold, since the sequence of partial sums of a double convergent series need not to be bounded.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz as follows

$$Z\left(\Delta\right)=\left\{x=\left(x_k\right)\in w:\left(\Delta x_k\right)\in Z\right\}$$

for $Z = c$, c_0 and l_{∞}, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in$ N.

Here, c, c_0 and l_{∞} denoted the classes of convergent, null and bounded sclar valued single sequences respectively. The difference sequence space bv_p of the classical space l_p is introduced and studied in the case $1 \leq p \leq \infty$ by BaSar and Altay and in the case $0 < p < 1$ by Altay and BaSar in [20]. The spaces $c(\Delta)$, $c_0(\Delta)$, $l_{\infty}(\Delta)$ and bv_p are Banach spaces normed by

$$\|x\| = |x_1| + sup_{k\geq 1}|\Delta x_k| \text{ and } \|x\|_{bv_p} = (\textstyle\sum_{k=1}^{\infty}|x_k|^p)^{\frac{1}{p}}, (1 \leq p \leq \infty)$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z\left(\Delta\right)=\left\{x=\left(x_{mn}\right)\in w^2:\left(\Delta x_{mn}\right)\in Z\right\}$$

where $Z = \Lambda^2$, χ^2 and $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m, n \in$N.

Definition And Preliminaries

Let $n \in \mathbb{N}$ and X be a real vector space of dimension w, where $n \leq w$. A real valued function $d_p\left(x_1,\ldots,x_n\right)=\left\|\left(d_1\left(x_1\right),\ldots,d_n\left(x_n\right)\right)\right\|_p$ on X satisfying the following four conditions:

(i) $\left\|\left(d_1\left(x_1\right),\ldots,d_n\left(x_n\right)\right)\right\|_p=0$ if and only if $d_1(x_1), \ldots, d_n(x_n)$ are linearly dependent,

(ii) $\left\|\left(d_1\left(x_1\right),\ldots,d_n\left(x_n\right)\right)\right\|_p$ is invariant under permutation,

(iii) $\left\|\alpha d_1(x_1),\ldots,d_n(x_n)\right\|_p=|\alpha|\left\|d_1(x_1),\ldots,d_n(x_n)\right\|,\alpha\in\mathbb{R}$

(iv) $d_p\left(\left(x_1,y_1\right),\left(x_2,y_2\right)\ldots\left(x_n,y_n\right)\right)=\left(d_X\left(x_1,x_2,\ldots,x_n\right)^p+d_Y\left(y_1,y_2,\ldots,y_n\right)^p\right)^{1/p}$ for $1 \leq p < \infty$;

(or)

(v) $d\left(\left(x_1,y_1\right),\left(x_2,y_2\right),\ldots,\left(x_n,y_n\right)\right):=\sup\left\{d_X\left(x_1,x_2,\ldots,x_n\right),d_Y\left(y_1,y_2,\ldots,y_n\right)\right\}$, for $x_1, x_2, \ldots, x_n \in X$, $y_1, y_2, \ldots, y_n \in Y$ is called the p product metric of the Cartesian product of n metric spaces is the p norm of the n-vector of the norms of the n sub-spaces.

A trivial example of p product metric of n metric space is the p norm space is X = R equipped with the following Euclidean metric in the product space is the p norm:

$$\left\|\left(d_1(x_1),\ldots,d_n(x_n)\right)\right\|_E = \sup\left(\left|\det\left(d_{mn}(x_{mn})\right)\right|\right) = \sup\left(\left|\begin{matrix} d_{11}(x_{11}) & d_{12}(x_{12}) & \ldots & d_{1n}(x_{1n}) \\ d_{21}(x_{21}) & d_{22}(x_{22}) & \ldots & d_{2n}(x_{1n}) \\ . & & & \\ . & & & \\ . & & & \\ d_{n1}(x_{n1}) & d_{n2}(x_{n2}) & \ldots & d_{nn}(x_{nn}) \end{matrix}\right|\right)$$

where $x_i = (x_{i1}, \ldots, x_{in}) \in R^n$ for each $i = 1, 2, \ldots n$.

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the p– metric. Any complete p– metric space is said to be p– Banach metric space.

Let X be a linear metric space. A function w : $X \to$ R is called paranorm, if

(1) $w(x) \geq 0$, for all $x \in X$;

(2) $w(-x) = w(x)$, for all $x \in X$;

(3) $w(x + y) \leq w(x) + w(y)$, for all $x, y \in X$;

(4) If (σ_{mn}) is a sequence of scalars with $\sigma_{mn} \to \sigma$ as $m, n \to \infty$ and (x_{mn}) is a sequence of vectors with $w(x_{mn} - x) \to 0$ as $m, n \to \infty$, then $w(\sigma_{mn}x_{mn} - \sigma x) \to 0$ as $m, n \to \infty$.

A paranorm w for which $w(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, w) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [32], Theorem 10.4.2, p. 183).

Let $f = (f_{mn})$ be a Musielak-modulus function, $\left(X, \left\|\left(d(x_1), d(x_2), \ldots, d(x_{n-1})\right)\right\|_p\right)$ be a p– metric space, $q = (q_{mn})$ be bounded sequence of strictly positive real numbers and $u = (u_{mn})$ be any sequence of strictly positive real numbers. By $S(p - X)$ we denote the space of all sequence defined over $\left(X, \left\|\left(d(x_1), d(x_2), \ldots, d(x_{n-1})\right)\right\|_p\right)$.

In the present paper we define the following sequence spaces:

$$\left[\Lambda^{2q}_{fcu}, \left\|\left(d(x_1), d(x_2), \ldots, d(x_{n-1})\right)\right\|_p\right] =$$

$$\left\{\lim_{\eta\mu} \frac{1}{\eta\mu} \sum_{m=1}^{\eta} \sum_{n=1}^{\mu} \left[f_{mn}\left(\left\| u_{mn} \left|x_{m+s,n+s}\right|^{1/m+n+2s}, \left(d(x_1), d(x_2), \ldots, d(x_{n-1})\right)\right\|_p\right)\right]^{q_{mn}} = 0, \right\}$$

uniformly in s,

$$\left[\Lambda^{2q}_{\hat{f}cu}, \left\|\left(d(x_1), d(x_2), \ldots, d(x_{n-1})\right)\right\|_p\right] =$$

$$\left(\sup_{s,\eta,\mu} \frac{1}{\eta\mu} \sum_{m=1}^{\eta} \sum_{n=1}^{\eta} \left[f_{mn}\left(\left\| u_{mn} \left|x_{m+s,n+s}\right|^{1/m+n+2s}, \left(d(x_1), d(x_2), \ldots, d(x_{n-1})\right)\right\|_p\right)\right]^{q_{mn}} < \infty \right),$$

uniformly in s.

If we take $f_{mn}(x) = x$, we get

$$\left[\Lambda^{2q}_{\hat{f}cu}, \left\|\left(d(x_1), d(x_2), \ldots, d(x_{n-1})\right)\right\|_p\right] =$$

$$\left\{\lim_{\eta\mu} \frac{1}{\eta\mu} \sum_{m=1}^{\eta} \sum_{n=1}^{\eta} \left[\left(\left\| u_{mn} \left|x_{m+s,n+s}\right|^{1/m+n+2s}, \left(d(x_1), d(x_2), \ldots, d(x_{n-1})\right)\right\|_p\right)\right]^{q_{mn}} = 0, \right\}$$

uniformly in s,

$$\left[\Lambda^{2q}_{\hat{f}cu}, \left\|\left(d(x_1), d(x_2), \ldots, d(x_{n-1})\right)\right\|_p\right] =$$

$$\left(\sup_{s,\eta,\mu}\frac{1}{\eta\mu}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[f_{mn}\left(\left\|u_{mn}\left|x_{m+x,n+s}\right|^{1/m+n+2s},\left(d\left(x_1\right),d\left(x_2\right),...,d\left(x_{n-1}\right)\right)\right\|_p\right)\right]<\infty\right)$$

uniformly in s.

The following inequality will be used throughout the paper. If $0 \le q_{mn} \le \sup q_{mn} = H$, $K = max\,(1, 2^{H-1})$ then

(2.1) $$\left|a_{mn}+b_{mn}\right|^{q_{mn}} \le K\left\{\left|a_{mn}\right|^{q_{mn}}+\left|b_{mn}\right|^{q_{mn}}\right\}$$

for all m, n and a_{mn}, $b_{mn} \in$ C. Also $\left|a\right|^{q_{mn}} \le \max\left(1,\left|a\right|^{H}\right)$ for all a ∈ C.

The main aim of this paper is to study some sequence spaces defined by a Musielak-modulus function over p–metric spaces also study some topological properties and some inclusion relations between these spaces.

3. Main Results

3.1. Theorem. Let $f = (f_{mn})$ be a Musielak-modulus function, $q = (q_{mn})$ be analytic sequence of positive real numbers and $u = (u_{mn})$ be any sequence of strictly positive real numbers. Then spaces $\left[\Gamma^{2q}_{f\hat{c}u},\left\|\left(d\left(x_1\right),d\left(x_2\right),...,d\left(x_{n-1}\right)\right)\right\|_p\right]$ and $\left[\Lambda^{2q}_{fcu},\left\|\left(d\left(x_1\right),d\left(x_2\right),\ldots,d\left(x_{n-1}\right)\right)\right\|_p\right]$ are liner spaces.

Proof: It is routine verification. Therefore the proof is omitted.

3.2. Theorem: Let $f = (f_{mn})$ be a Musielak-modulus function, $q = (q_{mn})$ be analytic sequence of positive real numbers and $u = (u_{mn})$ be any sequence of strictly positive real numbers. Then space $\left[\Gamma^{2q}_{f\hat{c}u},\left\|\left(d\left(x_1\right),d\left(x_2\right),...d\left(x_{n-1}\right)\right)\right\|_p\right]$ is a paranormed space with respect to the paranorm defined by

$$g(x) = \inf\left(\frac{1}{\eta\mu}\sum_{m=1}^{\eta}\sum_{n=1}^{\eta}\left[f_{mn}\left(\left\|u_{mn}\left|x_{m+s,n+s}\right|^{1/m+n+2s},d\left(x_1\right),d\left(x_2\right),...,d\left(x_{n-1}\right)\right\|_p\right)\right]^{q_{mn}}\right)^{1/H} \le 1,$$

where $H = \max\,(1, \sup_{mn} q_{mn} < \infty)$.

Proof: Clearly $g(x) \ge 0$ for $x=\left(x_{mn}\right)\in\left[\Gamma^{2q}_{f\hat{c}u},\left\|\left(d\left(x_1\right),d\left(x_2\right),...,d\left(x_{n-1}\right)\right)\right\|_p\right]$.

Since $f_{mn}(0) = 0$, we get $g(0) = 0$.

Conversely, suppose that $g(x) = 0$, then

$$\inf\left(\left(\frac{1}{\eta\mu}\sum_{m=1}^{\eta}\sum_{1}^{\mu}\left[f_{mn}\left(\left\|u_{mn}\left|x_{m+s,n+s}\right|^{\frac{1}{m+n+2s}},\left(d\left(x_1\right),d\left(x_2\right),\ldots,d\left(x_{n-1}\right)\right)\right\|_p\right)\right]^{q_{mn}}\right)^{\frac{1}{H}} \le 1\right) = 0$$ Suppose that $\left|x_{m+s,n+s}\right|^{1/m+n+2s} \ne 0$ for each m, $n \in$ N. This implies that $u_{mn}\left|x_{m+s,n+s}\right|^{1/m+n+2s} \ne 0$, for each m, n, $s \in$ N.

Then $\left\|u_{mn}\left|x_{m+s,n+s}\right|^{1/m+n+2s},\left(d\left(x_1\right),d\left(x_2\right),...,d\left(x_{n-1}\right)\right)\right\|_p \to \infty$. It follows that

$$\left(\left(\frac{1}{\eta\mu}\sum_{m=1}^{\eta}\sum_{1}^{\mu}\left[f_{mn}\left(\left\|u_{mn}\left|x_{m+s,n+s}\right|^{\frac{1}{m+n+2s}},\left(d\left(x_1\right),d\left(x_2\right),\ldots,d\left(x_{n-1}\right)\right)\right\|_p\right)\right]^{q_{mn}}\right)^{\frac{1}{H}} \le 1\right) \to \infty$$,which is a contradiction. Therefore $u_{mn}\left|x_{m+s,n+s}\right|^{1/m+n+2s} = 0$ for each m, n and thus $\left|x_{m+s,n+s}\right|^{1/m+n+2s} = 0$ for each m, n ∈ N.

Let

$$\left(\frac{1}{\eta\mu}\sum_{m=1}^{\eta}\sum_{1}^{\mu}\left[f_{mn}\left(\left\|u_{mn}\left|x_{m+s,n+s}\right|^{\frac{1}{m+n+2s}},\left(d(x_1),d(x_2),\ldots,d(x_{n-1})\right)\right\|_p\right)\right]^{q_{mn}}\right)^{\frac{1}{H}} \leq 1 \text{ and}$$

$$\left(\frac{1}{\eta\mu}\sum_{m=1}^{\eta}\sum_{1}^{\mu}\left[f_{mn}\left(\left\|u_{mn}\left|y_{m+s,n+s}\right|^{\frac{1}{m+n+2s}},\left(d(x_1),d(x_2),\ldots,d(x_{n-1})\right)\right\|_p\right)\right]^{q_{mn}}\right)^{\frac{1}{H}} \leq 1$$

Then by using Minkowski's inequality, we have

$$\left(\frac{1}{\eta\mu}\sum_{m=1}^{\eta}\sum_{1}^{\mu}\left[f_{mn}\left(\left\|u_{mn}\left|x_{m+s,n+s}+y_{m+s,n+s}\right|^{\frac{1}{m+n+2s}},\left(d(x_1),d(x_2),\ldots,d(x_{n-1})\right)\right\|_p\right)\right]^{q_{mn}}\right)^{\frac{1}{H}}$$

$$\leq\left(\frac{1}{\eta\mu}\sum_{m=1}^{\eta}\sum_{n=1}^{\mu}\left[f_{mn}\left(\left\|u_{mn}\left|x_{m+s,n+s}\right|^{1/m+n+2s},\left(d(x_1),d(x_2),\ldots,d(x_{n-1})\right)\right\|_p\right)\right]^{q_{mn}}\right)^{1/H}+$$

$$\left(\frac{1}{\eta\mu}\sum_{m=1}^{\eta}\sum_{n=1}^{\mu}\left[f_{mn}\left(\left\|u_{mn}\left|y_{m+s,n+s}\right|^{1/m+n+2s},\left(d(x_1),d(x_2),\ldots,d(x_{n-1})\right)\right\|_p\right)\right]^{q_{mn}}\right)^{1/H}\leq 1,$$

so we have $g(x+y)=\inf\left(\frac{1}{\eta\mu}\sum_{m=1}^{\eta}\sum_{n=1}^{\mu}\right.$

$$\left.\left[f_{mn}\left(\left\|\left(u_{mn}x_{m+s,n+s}+u_{mn}y_{m+s,n+s}\right)\right\|^{1/m+n+2s},\left(d(x_1),d(x_2),\ldots,d(x_{n-1})\right)\right\|_p\right)\right]^{q_{mn}}\right)^{1/H}\leq 1$$

$$\leq\inf\left(\left(\frac{1}{\eta\mu}\sum_{m=1}^{\eta}\sum_{n=1}^{\mu}\left[f_{mn}\left(\left\|u_{mn}\left|x_{m+s,n+s}\right|^{1/m+n+2s},\left(d(x_1),d(x_2),\ldots,d(x_{n-1})\right)\right\|_p\right)\right]^{q_{mn}}\right)^{1/H}\leq 1\right)+$$

$$\inf\left(\left(\frac{1}{\eta\mu}\sum_{m=1}^{\eta}\sum_{n=1}^{\mu}\left[f_{mn}\left(\left\|u_{mn}\left|y_{m+s,n+s}\right|^{1/m+n+2s},\left(d(x_1),d(x_2),\ldots,d(x_{n-1})\right)\right\|_p\right)\right]^{q_{mn}}\right)^{1/H}\leq 1\right).$$

Therefore, $g(x + y) \leq g(x) + g(y)$.

Finally, to prove that the scalar multiplication is continuous. Let λ be any complex number. By definition, $g(\lambda x) =$

$$\inf\left(\left(\frac{1}{\eta\mu}\sum_{m=1}^{\eta}\sum_{n=1}^{\mu}\left[f_{mn}\left(\left\|u_{mn}\left|x_{m+s,n+s}\right|^{1/m+n+2s},\left(d(x_1),d(x_2),\ldots,d(x_{n-1})\right)\right\|_p\right)\right]^{q_{mn}}\right)^{1/H}\leq 1\right).$$

Then

$f(\lambda x)=\inf\left(\left(|\lambda|t\right)^{q_{mn}/H}:\left(\frac{1}{\eta\mu}\sum_{m=1}^{\eta}\sum_{n=1}^{\mu}\right.\right.$

$$\left.\left.\left[f_{mn}\left(\left\|u_{mn}\left|x_{m+s,n+s}\right|^{1/m+n+2s},\left(d(x_1),d(x_2),\ldots,d(x_{n-1})\right)\right\|_p\right)\right]^{q_{mn}}\right)^{1/H}\leq 1\right).$$

where $t=\frac{1}{|\lambda|}$. Since $|\lambda|^{q_{mn}} \leq \max\left(1,|\lambda|^{\sup p_{mn}}\right)$, we have

$$g(\lambda x)\leq\max\left(1,|\lambda|^{\sup p_{mn}}\right)\inf\left(t^{q_{mn}/H} :\left(\frac{1}{\eta\mu}\sum_{m=1}^{\eta}\sum_{n=1}^{\mu}\right.\right.$$

$$\left.\left.\left[f_{mn}\left(\left\|u_{mn}\left|x_{m+s,n+s}\right|^{1/m+n+2s},\left(d(x_1),d(x_2),\dots,d(x_{n-1})\right)\right\|_p\right)\right]^{q_{mn}}\right)^{1/H}\leq 1\right).$$

This completes the proof.

3.3. Theorem Let $f = (f_{mn})$ be a Musielak-modulus function. Then the following statements are equivalent

(i) $\left[\Lambda_{\hat{cu}}^{2q},\left\|\left(d(x_1),d(x_2),\dots,d(x_{n-1})\right)\right\|_p\right]\subseteq\left[\Lambda_{f\hat{cu}}^{2q},\left\|\left(d(x_1),d(x_2),\dots,d(x_{n-1})\right)\right\|_p\right].$

(ii) $\left[\Gamma_{\hat{cu}}^{2q},\left\|\left(d(x_1),d(x_2),\dots,d(x_{n-1})\right)\right\|_p\right]\subseteq\left[\Gamma_{f\hat{cu}}^{2q},\left\|\left(d(x_1),d(x_2),\dots,d(x_{n-1})\right)\right\|_p\right].$

(iii) $\sup_n \frac{1}{\eta\mu}\sum_{m=1}^{\eta}\sum_{n=1}^{\mu}\left[f_{mn}\left(\left\|u_{mn}\left|x_{m+s,n+s}\right|^{1/m+n+2s},\left(d(x_1),d(x_2),\dots,d(x_{n-1})\right)\right\|_p\right)\right]^{q_{mn}}<\infty.$

Proof: (i) $\Rightarrow$ (ii) is obvious, since

$\left[\Gamma_{\hat{cu}}^{2q},\left\|\left(d(x_1),d(x_2),\dots,d(x_{n-1})\right)\right\|_p\right]\subseteq\left[\Lambda_{\hat{cu}}^{2q},\left\|\left(d(x_1),d(x_2),\dots,d(x_{n-1})\right)\right\|_p\right].$

(ii) $\Rightarrow$ (iii) Suppose

$\left[\Gamma_{\hat{cu}}^{2q},\left\|\left(d(x_1),d(x_2),\dots,d(x_{n-1})\right)\right\|_p\right]\subseteq\left[\Lambda_{f\hat{cu}}^{2q},\left\|\left(d(x_1),d(x_2),\dots,d(x_{n-1})\right)\right\|_p\right].$ and let (iii) does not hold. Then

$$\sup_{\eta\mu}\frac{1}{\eta\mu}\sum_{m=1}^{\eta}\sum_{n=1}^{\mu}\left[f_{mn}\left(\left\|u_{mn}\left|x_{m+s,n+s}\right|^{1/m+n+2s},\left(d(x_1),d(x_2),\dots,d(x_{n-1})\right)\right\|_p\right)\right]^{q_{mn}}=\infty$$ and

therefore there is a sequence $(\eta_i\mu_j)$ of positive integers such that (3.1)

$$\frac{1}{\eta_i\mu_j}\sum_{m=1}^{\eta_i}\sum_{n=1}^{\mu_j}\left[f_{mn}\left(\left\|u_{mn}(ij)^{-(m+n+2s)},\left(d(x_1),d(x_2),\dots,d(x_{n-1})\right)\right\|_p\right)\right]^{q_{mn}}>(ij)^{m+n+2s},$$

$i, j = 1, 2 \dots$

Define $x = (x_{mn})$ by

$$x=(x_{mn})=\begin{cases}(ij)^{-(m+n+2s)}, 1\leq m\leq\eta_i;\ 1\leq n\leq\mu_j, \text{ if } i,j=1,2,3,\dots; \\ 0, \qquad \text{if } m\geq\eta_i, n\geq\mu_j.\end{cases}$$

Then $x=(x_{mn})\in\left[\Gamma_{f\hat{cu}}^{2q},\left\|\left(d(x_1),d(x_2),\dots,d(x_{n-1})\right)\right\|_p\right]$ but

$x=(x_{mn})\notin\left[\Lambda_{f\hat{cu}}^{2q},\left\|\left(d(x_1),d(x_2),\dots,d(x_{n-1})\right)\right\|_p\right]$ which contradicts (ii). Hence (iii) must hold.

(iii) $\Rightarrow$ (i). Suppose $x=(x_{mn})\in\left[\Lambda_{\hat{cu}}^{2q},\left\|\left(d(x_1),d(x_2),\dots,d(x_{n-1})\right)\right\|_p\right]$ and

$x=(x_{mn})\notin\left[\Lambda_{f\hat{cu},}^{2q}\left\|\left(d(x_1),d(x_2),\dots,d(x_{n-1})\right)\right\|_p\right]$. Then (3.2)

$$\sup_{s\eta\mu}\frac{1}{\eta\mu}\sum_{m=1}^{\eta}\sum_{n=1}^{\mu}\left[f_{mn}\left(\left\|u_{mn}\left|x_{m+s,n+s}\right|^{1/m+n+2s},\left(d(x_1),d(x_2),\ldots,d(x_{n-1})\right)\right\|_p\right)\right]^{q_{mn}}=\infty.$$

which contradicts (iii). Hence (i) must hold.

3.4. Theorem. Let $1\le q_{mn}\le \sup_{mn}q_{mn}<\infty$. Then the following statements are equivalent

(i) $\left[\Gamma^{2q}_{f\hat{cu}},\left\|\left(d(x_1),d(x_2),\ldots,d(x_{n-1})\right)\right\|_p\right]\subseteq\left[\Gamma^{2q}_{\hat{cu}}\left\|\left(d(x_1),d(x_2),\ldots,d(x_{n-1})\right)\right\|_p\right]$,

(ii) $\left[\Gamma^{2q}_{f\hat{cu}},\left\|\left(d(x_1),d(x_2),\ldots,d(x_{n-1})\right)\right\|_p\right]\subseteq\left[\Lambda^{2q}_{\hat{cu}},\left\|\left(d(x_1),d(x_2),\ldots,d(x_{n-1})\right)\right\|_p\right]$,

(iii) $\inf_{\eta\mu}\frac{1}{\eta\mu}\sum_{m=1}^{\eta}\sum_{n=1}^{\mu}\left[f_{mn}\left(\left\|u_{mn}\left|x_{m+s,n+s}\right|^{1/m+n+2s},\left(d(x_1),d(x_2),\ldots,d(x_{n-1})\right)\right\|_p\right)\right]^{q_{mn}}>0, t>0.$

Proof: (i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (iii) Suppose

$\left[\Gamma^{2q}_{f\hat{cu}},\left\|\left(d(x_1),d(x_2),\ldots,d(x_{n-1})\right)\right\|_p\right]\subseteq\left[\Lambda^{2q}_{\hat{cu}},\left\|\left(d(x_1),d(x_2),\ldots,d(x_{n-1})\right)\right\|_p\right]$ and let (iii) does not hold. Then

$$(3.3)\quad \inf_{\eta\mu}\frac{1}{\eta\mu}\sum_{m=1}^{\eta}\sum_{n=1}^{\mu}\left[f_{mn}\left(\left\|u_{mn}\left|x_{m+s,n+s}\right|^{1/m+n+2s},\left(d(x_1),d(x_2),\ldots,d(x_{n-1})\right)\right\|_p\right)\right]^{q_{mn}}=0, t>0.$$

We can choose an index sequence $(\eta_i\mu_j)$ such that

$$\frac{1}{\eta_i\mu_j}\sum_{m=1}^{\eta_i}\sum_{n=1}^{\mu_j}\left[f_{mn}\left(\left\|u_{mn}(ij)^{(m+n+2s)},\left(d(x_1),d(x_2),\ldots,d(x_{n-1})\right)\right\|_p\right)\right]^{q_{mn}}>(ij)^{-(m+n+2s)},$$

$i, j = 1, 2, \ldots$

Define $x=(x_{mn})$ by

$$x=(x_{mn})=\begin{cases}(ij)^{(m+n+2s)}, 1\le m\le\eta_i; 1\le n\le\mu_j, \text{ if } i,j=1,2,3,\ldots;\\ 0, \qquad \text{if } m\ge\eta_i, n\ge\mu_j.\end{cases}$$

Thus by (3.3) we have $x=(x_{mn})\in\left[\Gamma^{2q}_{\hat{cu}},\left\|\left(d(x_1),d(x_2),\ldots,d(x_{n-1})\right)\right\|_p\right]$ but

$x=(x_{mn})\notin\left[\Lambda^{2q}_{\hat{cu}},\left\|\left(d(x_1),d(x_2),\ldots,d(x_{n-1})\right)\right\|_p\right]$ which contradicts (ii). Hence (iii) must hold.

(iii) $\Rightarrow$ (i). Let $x=(x_{mn})\in\left[\Gamma^{2q}_{f\hat{cu}},\left\|\left(d(x_1),d(x_2),\ldots,d(x_{n-1})\right)\right\|_p\right]$ That is,

$$(3.4)\ \inf_{\eta\mu}\frac{1}{\eta\mu}\sum_{m=1}^{\eta}\sum_{n=1}^{\mu}\left[f_{mn}\left(\left\|u_{mn}\left|x_{m+s,n+s}\right|^{1/m+n+2s},\left(d(x_1),d(x_2),\ldots,d(x_{n-1})\right)\right\|_p\right)\right]^{q_{mn}}=0,$$

uniformly in s. Suppose (iii) hold and $x=(x_{mn})\notin\left[\Gamma^{2q}_{\hat{cu}},\left\|\left(d(x_1),d(x_2),\ldots,d(x_{n-1})\right)\right\|_p\right]$. Then for some number $\varepsilon>0$ and index $\eta_0\mu_0$, we have

$\left[f_{mn}(\varepsilon_0)\right]^{q_{mn}}\le\left[f_{mn}\left(\left\|u_{mn}\left|x_{m+s,n+s}\right|^{1/m+n+2s},\left(d(x_1),d(x_2),\ldots,d(x_{n-1})\right)\right\|_p\right)\right]^{q_{mn}}$ and consequently (3.4)

$$\lim_{\eta\mu}\frac{1}{\eta\mu}\sum_{m=1}^{\eta}\sum_{n=1}^{\mu}\left[f_{mn}(\varepsilon_0)\right]^{q_{mn}}=0,$$

which contradicts (iii). Hence

$$\left[\Gamma^{2q}_{f\hat{cu}},\left\|\left(d\left(x_1\right),d\left(x_2\right),\ldots,d\left(x_{n-1}\right)\right)\right\|_p\right]\subseteq\left[\Gamma^{2q}_{\hat{cu}},\left\|\left(d\left(x_1\right),d\left(x_2\right),\ldots,d\left(x_{n-1}\right)\right)\right\|_p\right].$$

This completes the proof.

3.5. Theorem. Let $f=(f_{mn})$ be a Musielak-modulus function. Let $1\le q_{mn}\le \sup_{mn}q_{mn}<\infty$. Then

$\left[\Lambda^{2q}_{f\hat{cu}},\left\|\left(d\left(x_1\right),d\left(x_2\right),\ldots,d\left(x_{n-1}\right)\right)\right\|_p\right]\subseteq\left[\Gamma^{2q}_{\hat{cu}},\left\|\left(d\left(x_1\right),d\left(x_2\right),\ldots,d\left(x_{n-1}\right)\right)\right\|_p\right]$ hold if and only if

(3.5)

$$\lim_{\eta\mu}\frac{1}{\eta\mu}\sum_{m=1}^{\eta}\sum_{n=1}^{\mu}\left[f_{mn}\left(\left\|u_{mn}\left|x_{m+s,n+s}\right|^{1/m+n+2s},\left(d\left(x_1\right),d\left(x_2\right),\ldots,d\left(x_{n-1}\right)\right)\right\|_p\right)\right]^{q_{mn}}=\infty,t>0$$

Proof:

Suppose $\left[\Lambda^{2q}_{f\hat{cu}},\left\|\left(d\left(x_1\right),d\left(x_2\right),\ldots,d\left(x_{n-1}\right)\right)\right\|_p\right]\subseteq\left[\Gamma^{2q}_{\hat{cu}}\left\|\left(d\left(x_1\right),d\left(x_2\right),\ldots,d\left(x_{n-1}\right)\right)\right\|_p\right]$, and let (3.5) does not hold. There is a number $t_0>0$ and index sequence $(\eta_i\mu_j)$ such that

(3.6)

$$\frac{1}{\eta_i\mu_j}\sum_{m=1}^{\eta_i}\sum_{n=1}^{\mu_j}\left[f_{mn}\left(\left\|u_{mn}\left|x_{m+s,n+s}\right|^{1/m+n+2s},\left(d\left(x_1\right),d\left(x_2\right),\ldots,d\left(x_{n-1}\right)\right)\right\|_p\right)\right]^{q_{mn}}\le N<\infty,$$

$i=1,2,\ldots$.

Define $x=(x_{mn})$ by

$$x=\left(x_{mn}\right)=\begin{cases}\left(t_0\right)^{(m+n+2s)},1\le m\le\eta_i;1\le n\le\mu_j,\text{ if } i,j=1,2,3,\ldots,;\\ 0, \qquad \text{if } m\ge\eta_i,n\ge\mu_j.\end{cases}$$

Therefore, $x=\left(x_{mn}\right)\in\left[\Lambda^{2q}_{f\hat{cu}},\left\|\left(d\left(x_1\right),d\left(x_2\right),\ldots,d\left(x_{n-1}\right)\right)\right\|_p\right]$ but

$x=\left(x_{mn}\right)\notin\left[\Gamma^{2q}_{\hat{cu}},\left\|\left(d\left(x_1\right),d\left(x_2\right),\ldots,d\left(x_{n-1}\right)\right)\right\|_p\right]$. Hence (3.5) must hold.

Conversely, if $x=\left(x_{mn}\right)\in\left[\Lambda^{2q}_{f\hat{cu}},\left\|\left(d\left(x_1\right),d\left(x_2\right),\ldots,d\left(x_{n-1}\right)\right)\right\|_p\right]$, then for each s, η and μ

(3.7)

$$\frac{1}{\eta\mu}\sum_{m=1}^{\eta}\sum_{n=1}^{\mu}\left[f_{mn}\left(\left\|u_{mn}\left|x_{m+s,n+s}\right|^{1/m+n+2s},\left(d\left(x_1\right),d\left(x_2\right),\ldots,d\left(x_{n-1}\right)\right)\right\|_p\right)\right]^{q_{mn}}\le N<\infty.$$

Suppose that $x=\left(x_{mn}\right)\notin\left[\Gamma^{2q}_{\hat{cu}},\left\|\left(d\left(x_1\right),d\left(x_2\right),\ldots,d\left(x_{n-1}\right)\right)\right\|_p\right]$. Then for some number $\varepsilon_0>0$ there is a number s_0 we have

$\left[f_{mn}\left(\varepsilon_0\right)\right]^{q_{mn}}\le\left[f_{mn}\left(\left\|u_{mn}\left|x_{m+s,n+s}\right|^{1/m+n+2s},\left(d\left(x_1\right),d\left(x_2\right),\ldots,d\left(x_{n-1}\right)\right)\right\|_p\right)\right]^{q_{mn}}$ and hence for m, n and s we get

$$\frac{1}{\eta\mu}\sum_{m=1}^{\eta}\sum_{n=1}^{\mu}\left[f_{mn}\left(\varepsilon_0\right)\right]^{q_{mn}}\le N<\infty,$$

for some $N>0$, which contradicts (3.5). Hence $\left[\Lambda^{2q}_{f\hat{cu}},\left\|\left(d\left(x_1\right),d\left(x_2\right),\ldots,d\left(x_{n-1}\right)\right)\right\|_p\right]\subseteq\left[\Gamma^{2q}_{\hat{cu}},\left\|\left(d\left(x_1\right),d\left(x_2\right),\ldots,d\left(x_{n-1}\right)\right)\right\|_p\right]$. This completes the proof.

3.6. Theorem. Let $f = (f_{mn})$ be a Musielak-modulus function. Let $1 \le q_{mn} \le \sup_{mn} q_{mn} < \infty$. Then $\left[\Lambda^{2q}_{\hat{cu}}, \left\|\left(d(x_1), d(x_2), \ldots, d(x_{n-1})\right)\right\|_p\right] \subseteq \left[\Gamma^{2q}_{f\hat{cu}}, \left\|\left(d(x_1), d(x_2), \ldots, d(x_{n-1})\right)\right\|_p\right]$ hold if and only if

(3.8)

$$\lim_{\eta\mu} \frac{1}{\eta\mu} \sum_{m=1}^{\eta} \sum_{n=1}^{\mu} \left[f_{mn} \left(\left\| u_{mn} \left| x_{m+s,n+s} \right|^{1/m+n+2s}, \left(d(x_1), d(x_2), \ldots, d(x_{n-1})\right) \right\|_p \right) \right]^{q_{mn}} = 0.$$

Proof: It is similar to above. Therefore we omit the proof.

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