ON COMPLETE, STRONG, REGULAR FUZZY GRAPHS

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Abstract: The fuzzy operations were discussed by Rosenfeld and Bhattacharya. The degree and total degree of fuzzy regular graphs have been studied by Nagoorgani and Radha. In this paper, we proved that the join, union, composition, cartesian and tensor product of two complete regular fuzzy graphs is also a complete strong regular fuzzy graphs provided, if necessary some conditions are satisfied and found the degree and total degree of them.

Keywords: Complete fuzzy graph, Strong fuzzy graph, Regular fuzzy graph.

Introduction: Zadeh[1] has defined a fuzzy set on a continuum of grades ranging between zero and one characterized by a membership(characteristic) and also introduced the notions of inclusion, join, complement, etc in his research paper. Later on in 1973, the basic idea of fuzzy graphs was introduced by Kauffman[2]. The concept of fuzzy graphs and its relations were elaborately discussed by Rosenfeld[3] in 1975. He obtained many theoretic analogs between graph theory and fuzzy graphs. Mordeson and Peng[4] have given the operations on fuzzy graphs. Nagoorgani and Radha[5,6] have discussed the regular fuzzy graphs and their degree and total degree. Fuzzy graphs has attracted many researchers in recent days.

In this paper, we considered the join, union, composition, cartesian and tensor products of two complete, strong and regular fuzzy graphs, if necessary with some conditions and proved the resulting graph is a complete strong regular fuzzy graphs for each case.

Definition 1.1 Let V be a non-empty set. A fuzzy graph G is a pair of functions $G:(\sigma,\mu)$ where σ is a fuzzy subset of a non-empty set V and μ is a symmetric fuzzy relation on σ.i.e., $\sigma: V \to [0,1]$ and $\mu: V\times V \to [0,1]$ such that $\mu(uv) = \sigma(u) \wedge \sigma(v)$ for all u, v in .

Definition 1.2 A fuzzy graph $G:(\sigma,\mu)$is strong if $\mu(uv) = \sigma(u) \wedge \sigma(v)\ \ \forall (u,v) \in \mu^*$ and is complete if $\mu(uv) = \sigma(u) \wedge \sigma(v),\ \ \forall (u,v) \in \sigma^*$ where $\sigma^* = supp(\sigma) = \{u \in V/\sigma(u) > 0\}$ and $\mu^* = supp(\mu) = \{(u,v) \in V \times V/\mu(u,v) > 0\}$.

Definition 1.3 Let $G:(\sigma,\mu)$ be a fuzzy graph. The degree of a vertex u is $d_G(u) = \sum_{u\neq v} \mu(uv)$. Since $\mu(uv) > 0$ for $uv \in E$ and $\mu(uv) = 0$ for $uv \notin E$, this is equivalent to $d_G(u) = \sum_{uv\in E} \mu(uv)$. The minimum degree of G is $\delta(G) = \wedge\{d(v) /v \in V\}$. The maximum degree of G is $\Delta(G) = \vee \{d(v)/v \in V\}$.

Definition 1.4 Let $G:(\sigma,\mu)$ be a fuzzy graph on $G^*:(V,E)$. If $d_G(v) = k$ for all $v \in V$, ($i.e.$) if each vertex has same degree k, then G is said to be a regular fuzzy graph of degree k or a k-regular fuzzy graph.

Definition 1.5 Let $G:(\sigma,\mu)$ be a fuzzy graph on G^*. The total degree of a vertex $u \in V$ is defined by $td_G(u) = \sum_{u\neq v} \mu(uv) + \sigma(u) = \sum_{uv\in E} \mu(uv) + \sigma(u) = d_G(u) + \sigma(u)$. If each vertex of G has the same total degree k, then G is said to be a totally regular fuzzy graph of total degree k or a k-totally regular fuzzy graph.

Definition 1.6 The Union of two fuzzy graphs G_1 and G_2 is defined as a fuzzy graph $G = G_1 \cup G_2:(\sigma_1\cup\sigma_2, \mu_1\cup\mu_2)$ on $G^*:(V,E)$ where $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$ with

$$(\sigma_1\cup\sigma_2)(u) = \begin{cases} \sigma_1(u), & if\ u \in V_1 - V_2 \\ \sigma_2(u), & if\ u \in V_2 - V_1 \\ \sigma_1(u)\vee\sigma_2(u), & if\ u \in V_1 \cap V_2 \end{cases}$$

and

$$(\mu_1\cup\mu_2)(e) = \begin{cases} \mu_1(e), & if\ e \in E_1 - E_2 \\ \mu_2(e), & if\ e \in E_2 - E_1 \\ \mu_1(e)\vee\mu_2(e), & if\ e \in E_1 \cap E_2 \end{cases}$$

Definition 1.7 Assume that $V_1 \cap V_2 = \emptyset$. The join (sum) of G_1 and G_2 is defined as a fuzzy graph $G = G_1 + G_2 : (\sigma_1+\sigma_2, \mu_1+\mu_2)$ on $G^* : (V,E)$ where $V = V_1 \cup V_2$ and $E = E_1 \cup E_2 \cup E'$ where E' is the set of all edges joining vertices of V_1 with vertices of V_2, with $(\sigma_1+\sigma_2)(u) = (\sigma_1\cup\sigma_2)(u)$ for all $u \in V_1 \cup V_2$ and

$$(\mu_1+\mu_2)(uv) = \begin{cases} (\mu_1\cup\mu_2)(uv), & if\ uv \in E_1 \cup E_2 \\ \sigma_1(u)\wedge\sigma_2(v), & if\ uv \in E' \end{cases}$$

Definition 1.8 The composition of two fuzzy graphs G_1 and G_2 is defined as a fuzzy graph $G = G1[G2] : (\sigma_1\circ\sigma_2, \mu_1\circ\mu_2)$ on $G^* : (V,E)$ where $V = V_1 \times V_2$ and $E = \{((u_1,u_2)(v_1,v_2))/u_1 = v_1, u_2v_2 \in E_2\ or\ u_2 = v_2, u_1v_1 \in E_1\ or\ u_2 \neq v_2, u_1v_1 \in E_1\}$ with $(\sigma_1\circ\sigma_2)(u_1,u_2) = \sigma_1(u_1)\wedge\sigma_2(u_2)$ for all $(u_1,u_2) \in V_1 \times V_2$ and

$$(\mu_1\circ\mu_2)((u_1,u_2),(v_1,v_2)) = \begin{cases} \sigma_1(u_1)\wedge\mu_2(u_2v_2), if\ u_1 = v_1\ and\ u_2v_2 \in E_2 \\ \sigma_2(u_2)\wedge\mu_1(u_1v_1), if\ u_2 = v_2\ and\ u_1v_1 \in E_1 \\ \sigma_2(u_2)\wedge\sigma_2(v_2)\wedge\mu_1(u_1v_1), if\ u_2 \neq v_2\ and\ u_1v_1 \in E_1 \end{cases}$$

Definition 1.9 The cartesian product of two fuzzy graphs G_1 and G_2 is defined as a fuzzy graph $G = G_1 \times G_2:(\sigma_1\times\sigma_2, \mu_1\times\mu_2)$ on $G^* : (V,E)$ where $V = V_1 \times V_2$ and $E =$

$\{((u_1, u_2)(v_1, v_2))/u_1 = v_1, u_2v_2 \in E_2 \text{ or } u_2 = v_2, u_1v_1 \in E_1\}$, with $(\sigma_1 \times \sigma_2)(u_1, u_2) = \sigma_1(u_1) \wedge \sigma_2(u_2)$ for all $(u_1, u_2) \in V_1 \times V_2$ and $(\mu_1 \times \mu_2)((u_1, u_2)(v_1, v_2)) =$

$$\begin{cases} \sigma_1(u_1) \wedge \mu_2(u_2v_2), \text{if } u_1 = v_1 \text{ and } u_2v_2 \in E_2 \\ \sigma_2(u_2) \wedge \mu_1(u_1v_1), \text{if } u_2 = v_2 \text{ and } u_1v_1 \in E_1 \end{cases}$$

Definition 1.10 The tensor product of two fuzzy graphs (σ_i, μ_i) on $G_i = (V_i, X_i), i = 1, 2$ is defined as a fuzzy graph $G = G_1 \otimes G_2: (\sigma_1 \otimes \sigma_2, \mu_1 \otimes \mu_2)$ on $G = (V, X)$ where $V = V_1 \times V_2$ and $X = \{(u_1, u_2)(v_1, v_2)/(u_1, v_1) \in X_1, (u_2, v_2) \in X_2\}$. Fuzzy sets $\sigma_1 \otimes \sigma_2$ and $\mu_1 \otimes \mu_2$ are defined as $(\sigma_1 \otimes \sigma_2)(u_1, u_2) = \{\sigma_1(u_1) \wedge \sigma_2(u_2)\}$ for all $(u_1, u_2) \in V$

$(\mu_1 \otimes \mu_2)\{(u_1, u_2)(v_1, v_2)\} = \{\mu_1(u_1, v_1) \wedge \mu_2(u_2, v_2)\}$ for all $(u_1, u_2) \in X_1$ and $(v_1, v_2) \in X_2$.

2.Main Results:

Theorem 2.1 The join of two complete strong totally regular fuzzy graphs G_1 and G_2 with $\sigma_1 = \sigma_2$ and $\mu_1 = \mu_2$ is a complete strong totally regular fuzzy graph.

Proof: Let $u_1, u_2, \ldots, u_m$ and $v_1, v_2, \ldots v_n$ be the vertices of the complete strong totally regular fuzzy graphs $G_1$and G_2 respectively.

By definition, $\sigma_1: u_i \to [0,1]$ and $\mu_1: (u_i, u_j) \to [0,1]$ for $i \neq j$ such that $\mu_1(u_i, u_j) = \sigma_1(u_i) \wedge \sigma_1(u_j)$ for all $u_i, u_j \in \sigma_1^*$ and $(u_i, u_j) \in \mu_1^*$ where $\sigma_1^* = supp(\sigma_1) = \{u_i \in V_1/\sigma_1(u_i) > 0\}$ and $\mu_1^* = supp(\mu_1) = \{(u_i, u_j) \in V_1 \times V_1 / \mu_1(u_i, u_j) > 0\}$.

Similarly, $\sigma_2: v_i \to [0,1]$ and $\mu_2: (v_i, v_j) \to [0,1]$ for $i \neq j$ such that $\mu_2(v_i, v_j) = \sigma_2(v_i) \wedge \sigma_2(v_j)$ for all $v_i, v_j \in \sigma_2^*$ and $(v_i, v_j) \in \mu_2^*$ where $\sigma_2^* = supp(\sigma_2) = \{v_i \in V_2/\sigma_2(v_i) > 0\}$ and $\mu_2^* = supp(\mu_2) = \{(v_i, v_j) \in V_2 \times V_2 / \mu_2(v_i, v_j) > 0\}$.

For any vertex $u \in V_1(or\, V_2)$ we have $\sigma_1(u) = \sigma_2(u) = c$ for all $u \in V_1(or\, V_2)$, where $c \in [0, 1]$ is a fixed membership function for all u.

Now, E_1 is the edge set of G_1 and E_2 is the edge set of G_2 and E' is the edge set which contains all the edges having one vertex in V_1 and other vertex in V_2 then $\mu_1(u, v) = \mu_2(u, v) = c$ for all $(u, v) \in E_1 \text{ or } E_2 \text{ or } E'$ where $c \in [0, 1]$. From these two conditions, it follows that the join of two complete strong totally regular fuzzy graphs is again a complete strong totally regular fuzzy graph denoted by $G_1 + G_2$.

The degree of a vertex in $G_1 + G_2$:

If $u \in V_1$, then

$$\boldsymbol{d_{G_1+G_2}(u)} = \sum_{uv \in E_1} \mu_1(uv) + \sum_{uv \in E'} \mu(uv)$$

If $u \in V_2$, then

$$\boldsymbol{d_{G_1+G_2}(u)} = \sum_{uv \in E_2} \mu_2(uv) + \sum_{uv \in E'} \mu(uv)$$

The total degree of a vertex in $G_1 + G_2$:

For any $u \in V_1$,

$\boldsymbol{td_{G_1+G_2}(u)} = \boldsymbol{d_{G_1+G_2}(u)} + \sigma(u) \text{ for all } u \in V_1$.

For any $v \in V_2$,

$\boldsymbol{td_{G_1+G_2}(v)} = \boldsymbol{d_{G_1+G_2}(v)} + \sigma(v) \text{ for all } v \in V_2$.

The join of G_1 and G_2, that is $G_1 + G_2$ has $m + n$ vertices and $\frac{(m+n)(m+n-1)}{2}$ edges in $G_1 + G_2$. If $w_1, w_2, \ldots, w_{m+n}$ are the vertices of the join of G_1 and G_2 that is $G_1 + G_2$, then $\sigma(w_i) = c$ for all $i = 1, 2, \ldots, m + n$ and $\mu(w_i, w_j) = c$ for all $i, j = 1, 2, \ldots m + n$. This shows that $G_1 + G_2$ is a complete strong totally regular fuzzy graph.

Remark : If the conditions $\sigma_1 = \sigma_2$ and $\mu_1 = \mu_2$ does not hold, then $G_1 + G_2$ is an irregular fuzzy graph.

Illustration 2.2

Theorem 2.3 The union of two complete strong regular fuzzy graphs G_1 and G_2 is either G_1 or G_2 depending on $V(G_1) > V(G_2)$ with $\sigma_1 > \sigma_2$ and $\mu_1 > \mu_2$ or $V(G_1) < V(G_2)$ with $\sigma_1 < \sigma_2$ and $\mu_1 < \mu_2$.

Proof: Let $u_1, u_2, \ldots, u_m$ and $u_1, u_2, \ldots, u_n$ be the vertices of two complete strong regular fuzzy graphs G_1 and G_2 respectively. Let e_1 and e_2 be the number of edges in G_1 and G_2 respectively.

By definition, $\sigma_1: u_i \to [0,1], i = 1, 2, \ldots, m$ and $\mu_1: (u_i, u_j) \to [0,1]$ for $i \neq j$ such that $\mu_1(u_i, u_j) = \sigma_1(u_i) \wedge \sigma_1(u_j)$ for all $u_i, u_j \in \sigma_1^*$ and $(u_i, u_j) \in \mu_1^*$ where $\sigma_1^* = supp(\sigma_1) = \{u_i \in V_1/\sigma_1(u_i) > 0\}$ and $\mu_1^* = supp(\mu_1) = \{(u_i, u_j) \in V_1 \times V_1 / \mu_1(u_i, u_j) > 0\}$.

Similarly, $\sigma_2: u_i \to [0,1], i = 1, 2, \ldots, n$ and $\mu_2: (u_i, u_j) \to [0,1]$ for $i \neq j$ such that $\mu_2(u_i, u_j) = \sigma_2(u_i) \wedge \sigma_2(u_j)$ for all $u_i, u_j \in \sigma_2^*$ and $(u_i, u_j) \in \mu_2^*$ where $\sigma_2^* = supp(\sigma_2) = \{u_i \in V_2/\sigma_2(u_i) > 0\}$ and $\mu_2^* = supp(\mu_2) = \{(u_i, u_j) \in V_2 \times V_2 / \mu_2(u_i, u_j) > 0\}$.

We discuss the cases $V(G_1) > V(G_2)$ and $V(G_1) < V(G_2)$ seperately as follows:

Case 1: $V(G_1) > V(G_2)$

The condition is that $\sigma_1(u_i) > \sigma_2(u_i)$ for all i and $\mu_1(u_i, u_j) > \mu_2(u_i, u_j)$ for all i, j. The union of G_1 and G_2 say $G_1 \cup G_2$ has the vertex set of G_1 and the vertices of G_2 are identified with the vertices of G_1. Without loss of generality, the vertices may be named as $u_1, u_2, \ldots, u_m$ in $G_1 \cup G_2$. For any $u_i \in V(G_1 \cup G_2)$ implies that $u_i \in V(G_1)$ and hence $\sigma(u_i) = \sigma_1(u_i)$ where σ is the membership function of $G_1 \cup G_2$ defined for vertices $u_i \in V(G_1)$ in $G_1 \cup G_2$. $\mu(u_i, u_j) = \mu_1(u_i, u_j)$ for all $(u_i, u_j) \in V(G_1 \cup G_2)$ and hence $(u_i, u_j) \in E(G_1)$. Where μ is a symmetric fuzzy relation on σ.

Case 2: $V(G_1) < V(G_2)$

A similar procedure may be followed for this case.

The degree of a vertex in $G_1 \cup G_2$:

1. If $V(G_1) > V(G_2)$ then $\boldsymbol{d_{G_1 \cup G_2}(u)} = \boldsymbol{d_{G_1}}(u)$ for $u \in V_1(or\, V_1 \cap V_2)$.
2. If $V(G_1) < V(G_2)$ then $\boldsymbol{d_{G_1 \cup G_2}(v)} = \boldsymbol{d_{G_2}(v)}$ for $v \in V_2(or\, V_1 \cap V_2)$.

The total degree of a vertex in $G_1 \cup G_2$:

$\boldsymbol{td_{G_1 \cup G_2}}(u) = \boldsymbol{d_{G_1 \cup G_2}}(u) + \sigma(u)$ for $u \in V_1$

$(or V_2 or\ V_1 \cap V_2)$.

Remark : If the conditions $\sigma_1 > \sigma_2$ and $\mu_1 > \mu_2$ for $V(G_1) > V(G_2)$ does not hold, then $G_1 \cup G_2$ is an irregular fuzzy graph.

Illustration 2.4

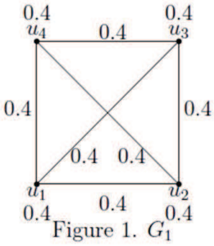

Figure 1. G_1

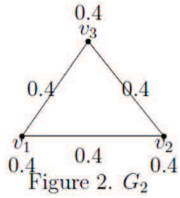

Figure 2. G_2

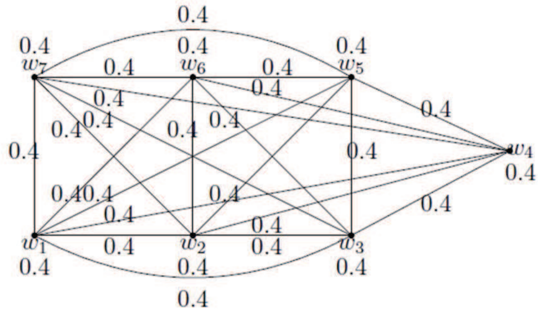

Figure 3. $G_1 + G_2$

$d_{G_1+G_2}(w_i) = 2.4$ for all $i = 1, 2, \ldots, m+n$.
$td_{G_1+G_2}(w_i) = 2.8$ for all $i = 1, 2, \ldots, m+n$.

Theorem 2.5 The composition of two complete strong regular fuzzy graphs $G_1: (\sigma_1, \mu_1)$ and $G_2: (\sigma_2, \mu_2)$ is a complete strong totally regular fuzzy graph if

(i) either σ_1 or σ_2 is a constant and

(ii) if σ_1 is a constant then $\sigma_1(u_i) < \sigma_2(v_j)$ for all $u_i \in V(G_1)$ and $v_j \in V(G_2)$ or if σ_2 is a constant then $\sigma_1(u_i) > \sigma_2(v_j)$ for all i, j.

Proof: Let $u_1, u_2, \ldots, u_m$ and $v_1, v_2, \ldots, v_n$ be the vertices of two complete strong regular fuzzy graphs G_1 and G_2 respectively. Let us suppose that

If σ_1 is constant then $\sigma_1(u_i) < \sigma_2(v_j)$ for $u_i \in V_1$ and $v_j \in V_2$.

If σ_2 is constant then $\sigma_1(u_i) > \sigma_2(v_j)$ for $u_i \in V_1$ and $v_j \in V_2$.

Case 1: Let σ_1 be a constant say c. Then

$\sigma_1(u_i) = c$ where $c \in [0,1]$

$\mu_1(u_i, u_j) = c$ where $c \in [0,1]$

such that $\mu_1(u_i, u_j) = \sigma_1(u_i) \wedge \sigma_1(u_j)$

For the fuzzy graph G_2, we have

$\sigma_2: v_i \to [0,1]$ and $\mu_2: (v_i, v_j) \to [0,1]$ for $i \neq j$ such that $\mu_2(v_i, v_j) = \sigma_2(v_i) \wedge \sigma_2(v_j)$ for all $v_i, v_j \in \sigma_2^*$ and $(v_i, v_j) \in \mu_2^*$ where $\sigma_2^* = supp(\sigma_2) = \{v_i \in V_2 / \sigma_2(v_i) > 0\}$ and $\mu_2^* = supp(\mu_2) = \{(v_i, v_j) \in V_2 \times V_2 / \mu_2(v_i, v_j) > 0\}$.

The composition of G_1 and G_2 denoted as $G_1[G_2]$ has the vertex set (u_i, v_j), $u_i \in V(G_1)$ and $v_j \in V(G_2)$. Any edge of $G_1[G_2]$ is denoted by $((u_i, v_j), (u_k, v_l))$ where $1 \leq i, k \leq m$ and $1 \leq j, l \leq n$. If σ and μ are the membership functions of $G_1[G_2]$, then $\sigma(u_i, v_j) = c$ for all $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$ and $\mu\left((u_i, v_j), (u_k, v_l)\right) = c$ for all $1 \leq i, k \leq m$ and $1 \leq j, l \leq n$. The number of vertices is mn and the number of edges is $\frac{mn(mn-1)}{2}$ in $G_1[G_2]$. Thus $G_1[G_2]$ is a complete strong totally regular fuzzy graph since c is a constant.

Case 2: For σ_2 constant, the same type of arguments holds.

The degree of a vertex in $G_1[G_2]$:

For a vertex $(u_1, v_1) \in V_1 \times V_2$, we have

$$d_{G_1[G_2]}(u_1, v_1) = \sum_{\substack{u_1 = u_2 \\ v_1 v_2 \in E_2}} \mu_2(v_1 v_2) + \sum_{\substack{v_1 = v_2 \\ u_1 u_2 \in E_1}} \mu_1(u_1 u_2) + \sum_{\substack{v_1 \neq v_2 \\ u_1 u_2 \in E_1}} \mu_1(u_1 u_2)$$

The total degree of a vertex in $G_1[G_2]$:

For a vertex $(u_1, v_1) \in V_1 \times V_2$ in $G_1[G_2]$,

$td_{G_1[G_2]}(u_1, v_1) = d_{G_1[G_2]}(u_1, v_1) + \sigma(u_1, v_1)$

Illustration 2.6

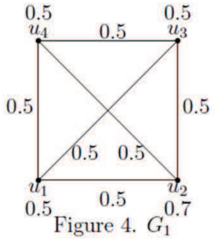

Figure 4. G_1

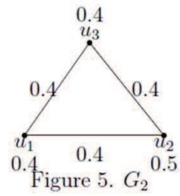

Figure 5. G_2

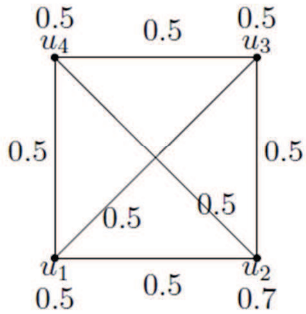

Figure 6. $G_1 \cup G_2$

$d_{G_1 \cup G_2}(u_1) = 1.5$, $td_{G_1 \cup G_2}(u_1) = 2.0$
$d_{G_1 \cup G_2}(u_2) = 1.5$, $td_{G_1 \cup G_2}(u_2) = 2.2$
$d_{G_1 \cup G_2}(u_3) = 1.5$, $td_{G_1 \cup G_2}(u_3) = 2.0$
$d_{G_1 \cup G_2}(u_4) = 1.5$, $td_{G_1 \cup G_2}(u_4) = 2.0$

Theorem 2.7 The Cartesian product of two complete strong regular fuzzy graphs $G_1(\sigma_1, \mu_1)$ and $G_2(\sigma_2, \mu_2)$ is a complete strong totally regular fuzzy graph if

(i) either σ_1 or σ_2 is a constant and

(ii) if σ_1 is a constant then $\sigma_1(u_i) < \sigma_2(v_j)$ for all $u_i \in V(G_1)$ and $v_j \in V(G_2)$ or if σ_2 is a constant then $\sigma_1(u_i) > \sigma_2(v_j)$ for all i, j.

Proof: Let $u_1, u_2, \ldots, u_m$ and $v_1, v_2, \ldots, v_n$ be the vertices of two complete strong regular fuzzy graphs

G_1 and G_2 respectively. Let us suppose that
If σ_1 is constant then $\sigma_1(u_i) < \sigma_2(v_j)$ for $u_i \in V_1$ and $v_j \in V_2$.
If σ_2 is constant then $\sigma_1(u_i) > \sigma_2(v_j)$ for $u_i \in V_1$ and $v_j \in V_2$.

Case 1: Let σ_1 be a constant say c. Then
$\sigma_1(u_i) = c$ where $c \in [0,1]$
$\mu_1(u_i, u_j) = c$ where $c \in [0,1]$
such that $\mu_1(u_i, u_j) = \sigma_1(u_i) \wedge \sigma_1(u_j)$
For the fuzzy graph G_2, we have
$\sigma_2: v_i \to [0,1]$ and $\mu_2: (v_i, v_j) \to [0,1]$ for $i \neq j$ such that $\mu_2(v_i, v_j) = \sigma_2(v_i) \wedge \sigma_2(v_j)$ for all $v_i, v_j \in \sigma_2^*$ and $(v_i, v_j) \in \mu_2^*$ where $\sigma_2^* = supp(\sigma_2) = \{v_i \in V_2 / \sigma_2(v_i) > 0\}$ and $\mu_2^* = supp(\mu_2) = \{(v_i, v_j) \in V_2 \times V_2 \,/\, \mu_2(v_i, v_j) > 0\}$.
The Cartesian product of G_1 and G_2 denoted as $G_1 \times G_2$ has the vertex set (u_i, v_j), $u_i \in V(G_1)$ and $v_j \in V(G_2)$. Any edge of $G_1 \times G_2$ is denoted by $((u_i, v_j), (u_k, v_l))$ where $1 \leq i, k \leq m$ and $1 \leq j, l \leq n$. If σ and μ are the membership functions of $G_1 \times G_2$, then $\sigma(u_i, v_j) = c$ for all $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$ and $\mu\left((u_i, v_j), (u_k, v_l)\right) = c$ for all $1 \leq i, k \leq m$ and $1 \leq j, l \leq n$. The number of vertices is mn and the number of edges is $\frac{mn(m+n-2)}{2}$ in $G_1 \times G_2$. Thus $G_1 \times G_2$ is a complete strong totally regular fuzzy graph since c is a constant.

Case 2: For σ_2 constant, the same type of arguments holds.

Illustration 2.8

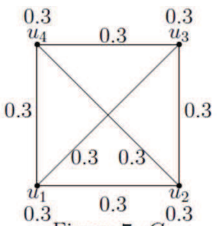

Figure 7. G_1

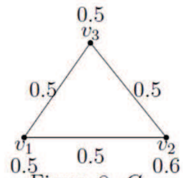

Figure 8. G_2

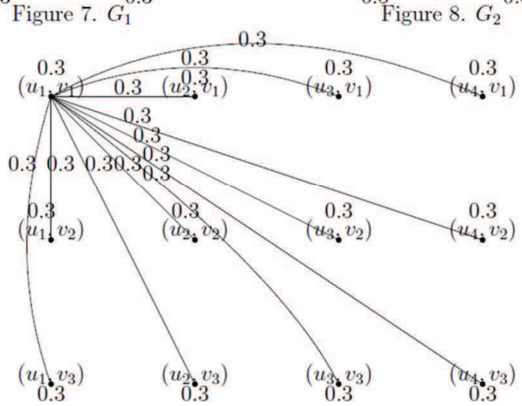
Figure 9. $G_1[G_2]$

$\sigma(u_i, v_j) = 0.3$ for $1 \leq i \leq m$ and $1 \leq j \leq n$.
$\mu((u_i, v_j), (u_k, v_l)) = 0.3$ for $1 \leq i, k \leq m$ and $1 \leq j, l \leq n$.
$d_{G_1[G_2]}(u_i, v_j) = 3.3$
$td_{G_1[G_2]}(u_i, v_j) = 3.6$

The degree of a vertex in $G_1 \times G_2$:
For a vertex $(u_1, v_1) \in V_1 \times V_2$, we have

$$d_{G_1 \times G_2}(u_1, v_1) = \sum_{\substack{u_1 = u_2 \\ v_1 v_2 \in E_2}} \mu_2(v_1 v_2) + \sum_{\substack{v_1 = v_2 \\ u_1 u_2 \in E_1}} \mu_1(u_1 u_2)$$

The total degree of a vertex in $G_1 \times G_2$:
For a vertex $(u_1, v_1) \in V_1 \times V_2$ in $G_1 \times G_2$,
$td_{G_1 \times G_2}(u_1, v_1) = d_{G_1 \times G_2}(u_1, v_1) + \sigma(u_1, v_1)$

Theorem 2.9 The tensor product of two complete strong regular fuzzy graphs $G_1: (\sigma_1, \mu_1)$ and $G_2: (\sigma_2, \mu_2)$ is a complete strong totally regular fuzzy graph if
(i) either σ_1 or σ_2 is a constant and
(ii) if σ_1 is a constant then $\sigma_1(u_i) < \sigma_2(v_j)$ for all $u_i \in V(G_1)$ and $v_j \in V(G_2)$ or if σ_2 is a constant then $\sigma_1(u_i) > \sigma_2(v_j)$ for all i, j.

Proof: Let $u_1, u_2, \ldots, u_m$ and $v_1, v_2, \ldots, v_n$ be the vertices of two complete strong regular fuzzy graphs G_1 and G_2 respectively. Let us suppose that
If σ_1 is constant then $\sigma_1(u_i) < \sigma_2(v_j)$ for $u_i \in V_1$ and $v_j \in V_2$.
If σ_2 is constant then $\sigma_1(u_i) > \sigma_2(v_j)$ for $u_i \in V_1$ and $v_j \in V_2$.

Case 1: Let σ_1 be a constant say c. Then
$\sigma_1(u_i) = c$ where $c \in [0,1]$
$\mu_1(u_i, u_j) = c$ where $c \in [0,1]$
such that $\mu_1(u_i, u_j) = \sigma_1(u_i) \wedge \sigma_1(u_j)$
For the fuzzy graph G_2, we have
$\sigma_2: v_i \to [0,1]$ and $\mu_2: (v_i, v_j) \to [0,1]$ for $i \neq j$ such that $\mu_2(v_i, v_j) = \sigma_2(v_i) \wedge \sigma_2(v_j)$ for all $v_i, v_j \in \sigma_2^*$ and $(v_i, v_j) \in \mu_2^*$ where $\sigma_2^* = supp(\sigma_2) = \{v_i \in V_2 / \sigma_2(v_i) > 0\}$ and $\mu_2^* = supp(\mu_2) = \{(v_i, v_j) \in V_2 \times V_2 \,/\, \mu_2(v_i, v_j) > 0\}$.
The tensor product of G_1 and G_2 denoted as $G_1 \otimes G_2$ has the vertex set (u_i, v_j), $u_i \in V(G_1)$ and $v_j \in V(G_2)$. Any edge of $G_1 \otimes G_2$ is denoted by $((u_i, v_j), (u_k, v_l))$ where $1 \leq i, k \leq m$ and $1 \leq j, l \leq n$. If σ and μ are the membership functions of $G_1 \otimes G_2$, then $\sigma(u_i, v_j) = c$ for all $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$ and $\mu\left((u_i, v_j), (u_k, v_l)\right) = c$ for all $1 \leq i, k \leq m$ and $1 \leq j, l \leq n$. The number of vertices is mn and the number of edges is $\frac{mn(m-1)(n-1)}{2}$ in $G_1 \otimes G_2$. Thus $G_1 \otimes G_2$ is a complete strong totally regular fuzzy graph since c is a constant.

Case 2: For σ_2 constant, the same type of arguments holds.

The degree of a vertex in $G_1 \otimes G_2$:
For a vertex $(u_1, v_1) \in V_1 \times V_2$, we have

$$d_{G_1 \otimes G_2}(u_1, v_1) = \sum_{u_1 u_2 \in E_1} \mu_1(u_1 u_2) \wedge \mu_2(v_1 v_2)$$

The total degree of a vertex in $G_1 \otimes G_2$:
For a vertex $(u_1, v_1) \in V_1 \times V_2$ in $G_1 \otimes G_2$,
$td_{G_1 \otimes G_2}(u_1, v_1) = d_{G_1 \otimes G_2}(u_1, v_1) + \sigma(u_1, v_1)$

Illustration 2.10

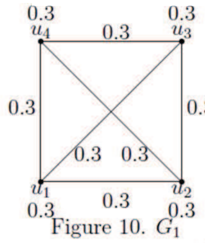

Figure 10. G_1

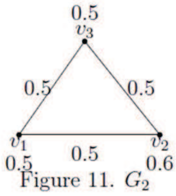

Figure 11. G_2

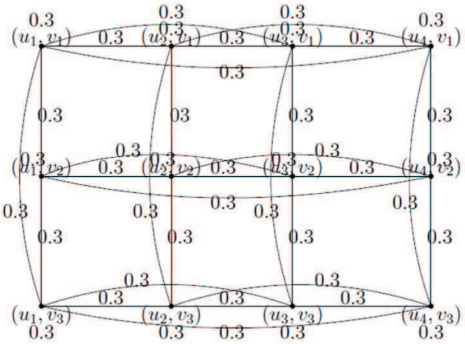

Figure 12. $G_1 \times G_2$

$\sigma(u_i, v_j) = 0.3$ for $1 \le i \le m$ and $1 \le j \le n$.
$\mu((u_i, v_j), (u_k, v_l)) = 0.3$ for $1 \le i, k \le m$ and $1 \le j, l \le n$.
$d_{G_1[G_2]}(u_i, v_j) = 1.5$
$td_{G_1[G_2]}(u_i, v_j) = 1.8$

3.Conclusion :

We have discussed the join and union of two complete strong regular fuzzy graphs. While considering the composition, cartesian and tensor product we have imposed the conditions either σ_1 or σ_2 is a constant and if σ_1 is constant, then $\sigma_1(u_i) < \sigma_2(v_j)$ for all $u_i \in V(G_1)$ and $v_j \in V(G_2)$ and if σ_2 is a constant then $\sigma_1(u_i) > \sigma_2(v_j)$ for all i, j. The resulting graph in all these three cases the number of vertices is mn and the number of edges varies. For composition, the number of edges in $G_1[G_2]$ is $\frac{mn(mn-1)}{2}$. For cartesian product, the number of edges in $G_1 \times G_2$ is $\frac{mn(m+n-2)}{2}$. For tensor product, the number of edges in $G_1 \otimes G_2$ is $\frac{mn(m-1)(n-1)}{2}$.

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