BOUNDS FOR EIGENVALUES OF SIMPLE GRAPHS

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Abstract : Let G be a simple graph. We give new upper and lower bounds for eigenvalues and spread of its adjacency matrix. We also consider the cases when the given graph is regular or bipartite.

Keywords: eigenvalues, graph, spread.

Introduction : Let G= (V, E) be a simple undirected graph with vertex set $V=\{v_1, v_2, \cdots, v_n\}$ of n vertices and edge set E of e edges. Let $d_i = d(v_i)$ be the degree of v_i. Without loss of generality we assume that $d_1 \geq \cdots \geq d_{i-1} \geq d_i \geq \cdots \geq d_n$. Let A be the n x n adjacency matrix of the graph G. All eigenvalues of A are real, as adjacency matrix of a simple graph is always real and symmetric. We denote them by $\lambda_i(G) = \lambda_i(A) = \lambda_i$ and assume that $\lambda_{\max} = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. We recall that spread, is given by $sp(G) = sp(A) = \lambda_1 - \lambda_n$.

We recall that chromatic number $\gamma(G)$ is k if G is k-colourable and not (k-1) –colourable. Bounds for $\gamma(G)$ can be obtained using bounds for largest eigenvalue (also called index) or spread, and λ_n, see [2], [3]. Also, let κ (G) be the size of largest clique in G: κ (G) is called the clique number of G. A bound for λ_1 can give a lower bound for κ (G), see [3].

By I_n we denote the identity matrix of order n and we represent the trace of A by tr A. Also transpose of matrix A is denoted by A^T. In section 2 first we present upper and lower bounds for the largest eigenvalue of a simple graph G. Bounds for regular and bipartite graphs are obtained as well. Also bounds for spread are derived. Finally two examples are presented in section 3.

We first give a lemma that will be used in our proof of Theorem 1.

Lemma 1 ([8], pp24)

Let B be a nonnegative n × n matrix with row sums $r_1, r_2, \cdots, r_n$. Then

$$\min_i r_i \leq \lambda_{\max}(B) \leq \max_i r_i .$$

When B is irreducible also, the equalities hold throughout if and only if all the row sums of B are all equal.

The above Lemma gives that

$$\lambda_{\max}(G) \leq d_1 . \qquad (1)$$

Also another bound is, see

$$\lambda_{\max}(G) \leq \sqrt{\frac{n-1}{n}\sum d_i} . \qquad (2)$$

Bounds For Eigenvalues : In the first two results we present bounds for the largest eigenvalue of a simple graph. Related work can be found in [7], [9]. The upper bound below is always at least as good as (1).

Theorem 1

Let G be a simple graph. Then

$$\lambda_{\max}(G) \leq \min_i \frac{d_i - 1 + \sqrt{(d_i+1)^2 + 4(i-1)(d_1 - d_i)}}{2} . \qquad (3)$$

Equality holds if G is a regular graph.

Proof: Let i =1 or $d_i = d_1$. Then (3) yields $\lambda_{\max}(G) \leq d_1$.

Now let $2 \leq i \leq n$ and $d_1 \geq \cdots \geq d_{i-1} > d_i \geq \cdots \geq d_n$.

We rewrite the adjacency matrix $A = (a_{ij})$ as,

$$A = \begin{pmatrix} P & R \\ R^T & Q \end{pmatrix},$$

Where P and Q are matrices of order (i -1) and (n-i+1), respectively. Further we assume that

$$U = \begin{pmatrix} xI_{i-1} & 0 \\ 0 & I_{n-i+1} \end{pmatrix},$$

where, x is a real number always greater than one. Then,

$$B = U^{-1} A U = \begin{pmatrix} P & \frac{1}{x}R \\ xR^T & Q \end{pmatrix}.$$

Clearly *A* and B have same eigenvalues. In particular, $\lambda_{\max}(A) = \lambda_{\max}(B)$. Let $r_j(B)$, $j = 1, \cdots, n$ be the row sums of the matrix B. Then we have:

for $1 \leq \ell \leq i-1$,

$$r_\ell(B) = \sum_{j=1}^{i-1} a_{\ell j} + \frac{1}{x}\sum_{j=i}^{n} a_{\ell j} = \frac{1}{x}\sum_{j=1}^{n} a_{\ell j} + (1 - \frac{1}{x})\sum_{j=1}^{i-1} a_{\ell j},$$

or $\quad r_\ell(B) = \frac{1}{x} d_\ell + (1-\frac{1}{x})\sum_{j=1}^{i-1} a_{\ell j},$

for $i \le k \le n$,

$$r_k(B) = x\sum_{j=1}^{i-1} a_{kj} + \sum_{j=i}^{n} a_{kj} = \sum_{j=1}^{n} a_{kj} + (x-1)\sum_{j=1}^{i-1} a_{kj}$$

$$= d_k + (x-1)\sum_{j=1}^{i-1} a_{kj}.$$

Since $x > 1$ and $d_1 \ge \cdots \ge d_{i-1} > d_i \ge \cdots \ge d_n$,

for $1 \le \ell \le i-1$,

$$r_\ell(B) \le \frac{1}{x} d_1 + (1-\frac{1}{x})\,(i-2),$$

and for $i \le k \le n$,

$$r_k(B) \le d_i + (x-1)(i-1).$$

Thus,

$$\max_j r_j(B) \le \max\{\frac{1}{x} d_1 + (1-\frac{1}{x})(i-2),\, d_i + (x-1)(i-1)\}.$$

We,set

$$\frac{1}{x} d_1 + (1-\frac{1}{x})\,(i-2) = d_i + (x-1)(i-1).$$

Solving for x,

$$x = \frac{2i-3-d_i+\sqrt{(d_i+1)^2+4(i-1)(d_1-d_i)}}{2(i-1)},$$

Since $i \ge 2$ and $d_1 > d_i$, we have $x > 1$. Hence by Lemma 1 we have,

$$\lambda_{\max}(A) = \lambda_{\max}(B) \le d_i + (x-1)\,(i-1).$$

Putting the value of x we have,

$$\lambda_{\max}(A) \le d_i + \left(\frac{2i-3-d_i+\sqrt{(d_i+1)^2+4(i-1)(d_1-d_i)}}{2(i-1)} - 1\right)(i-1).$$

or,

$$\lambda_{\max}(A) \le d_i + \left(\frac{-1-d_i+\sqrt{(d_i+1)^2+4(i-1)(d_1-d_i)}}{2(i-1)}\right)(i-1).$$

or,

$$\lambda_{\max}(A) \le \frac{d_i - 1 + \sqrt{(d_i+1)^2+4(i-1)(d_1-d_i)}}{2}.$$

Hence (3) follow. That equality holds when G is regular is clear.

In our next result we obtain two lower bounds for the largest eigenvalue of a simple graph.

Theorem 2

Let G be a simple graph with the adjacency matrix A of order n. Then,

$$\lambda_{\max}(G) \ge \frac{1}{\sqrt{2n}}(d_1+d_2). \qquad (4)$$

Also,

$$\lambda_{\max}(G) \ge \nu, \qquad (5)$$

where, $\quad m = \sum_i \frac{d_i}{n}$,

$$\nu^2 = \frac{1}{n}\sum_i d_i^2 - m^2. \qquad (6)$$

Proof: Since A is symmetric nonnegative,

$$\lambda_{\max}(G) = \max_{\|x\|=\|y\|=1} y^T A x,$$

Where, 2- norm is used (See [4, pp449]). Setting $x = \frac{1}{\sqrt{n}}(1,1,\cdots,1)^T$ and $y = \frac{1}{\sqrt{2}}(e_k+e_l), k \ne l,$ where e_i is the i^{th} column of the identity matrix I_n. Thus we have $\quad \lambda_{\max}(A) \ge y^T A x = \frac{1}{\sqrt{2n}}(d_k+d_l)$. Taking maximum over all k and $l, k \ne l$ (4) follows.

When we choose $x = \frac{1}{\sqrt{n}}(1,1,\cdots,1)^T$ and $y = (y_1, y_2, \cdots, y_n)^T$, where, $y_j = \frac{d_j - m}{\nu\sqrt{n}}$.

Then

$$\lambda_{\max}(G) \ge \frac{1}{\sqrt{n}}\sum y_k\, d_k$$

$$= \frac{1}{n\nu}\sum_k (d_k - m)\, d_k$$

$$\frac{1}{n\nu}(\sum d_k^2 - m\sum d_k) = \nu.$$

Hence (5) follow.

The next two results use ideas developed in [10].

Theorem 3

Let G be an r-regular graph with the adjacency matrix A of order $n \ge 2$. Then for $2 \le k \le n-1$,

$$-\left(r + \left(\frac{nr(k-1)(n-r-1)}{n-k}\right)^{1/2}\right) \le (n-1)\lambda_k \le -r + \left(\frac{nr(n-r-1)(n-k-1)}{k}\right)^{1/2}$$

(7)

Further,

$$(n-1)\,\lambda_n \geq -\min\{r(n-1),\, r+((n-2)(n-r-1)nr)^{1/2}\},$$
(8)

and

$$(n-1)\,\lambda_n \leq -\min\left\{n-1, r+\left(\frac{nr(n-r-1)}{n-2}\right)^{1/2}\right\}$$
(9)

Proof: We rewrite,

$\sum_i \lambda_i = 0$ and $\sum_i \lambda_i^2 = nr$,

as $\sum_{i=2}^{n} \lambda_i = -r$ and $\sum_{2}^{n} \lambda_i^2 = nr - r^2$, employing Theorem 2.2 of [10] we readily obtain the inequalities (7) . The inequality (8) follows from Theorem 2.1 of [10] and Perron Frobenious Theorem [2, pp18]. Finally inequality (9) follows from Theorem 2.1 of [10] and the interlacing inequality of eigenvalues of a real symmetric matrix, see [2, pp 21].

Our next result is for a simple bipartite graph. We assume that $G=(V\cup W, E)$ is a bipartite graph with two disjoint sets of vertices with cardinality m and n such that $m \leq n$. Specifically vertex sets V and W are, $V=\{v_1, v_2, \cdots, v_m\}, W=\{w_1, w_2, \cdots, w_n\}$. Let $v_1, v_2, \cdots, v_m$ have degrees $d_1 \geq \cdots \geq d_{i-1} \geq d_i \geq \cdots \geq d_m$, respectively and denote $e = d_1 + d_2 + \cdots + d_m$ .Then adjacency matrix of G is an $(m+n)\times(\mathrm{m}+\mathrm{n})$ matrix

$$A=\begin{pmatrix} 0 & P \\ P^T & 0 \end{pmatrix},$$

Where, P is a non-zero, 0-1 matrix of order $m \times n$ with $m \leq n$. Then

$$A^2=\begin{pmatrix} PP^T & 0 \\ 0 & P^T P \end{pmatrix}.$$

Let $D=PP^T=(d_{ij})$ and N_i be the set of neighbors of v_i . Then $d_{ij}=|N_i \cap N_j|$ is the cardinality of $N_i \cap N_j$ and $d_{ii}=d_i$, $1\leq i, j\leq m$. Also $tr\,D=e$ and $\mathrm{tr}\,\mathrm{D}^2=\sum_{i,j}^{m} d_{ij}^{\,2}=h$. We have the following theorem:

Theorem 4

Let G be a bipartite graph. Then

$$\sqrt{\frac{e}{m}+\sqrt{\frac{m-1}{m}(h-\frac{e^2}{m})}} \geq \lambda_{\max}(G) \geq \sqrt{\frac{e}{m}+\sqrt{\frac{1}{m(m-1)}(h-\frac{e^2}{m})}} \geq \sqrt{\frac{e}{m}}.$$
(10)

Equality will hold throughout if $d_1 = d_2 = \cdots = d_m = 1$.

Further, for $2\leq k\leq m$,

$$\lambda_k(G) \leq \sqrt{\frac{e}{m}+\sqrt{\frac{m-k}{mk}(h-\frac{e^2}{m})}}\ , \qquad (11)$$

and

$$\lambda_k(G) \geq \sqrt{\frac{e}{m}-\sqrt{\frac{k-1}{m(m-k+1)}(h-\frac{e^2}{m})}} \qquad (12)$$

Proof: The inequalities (10), (11) and (12) follow from Theorem 2.1 and Theorem 2.2 of [10]. Equality condition in (10) is clear.

Below we present a generalization of theorem 1.5 of [5]. Let be the n_0 number of non-isolated vertices of G and let G_0 denote the subgraph of order n_0 obtained by deleting the isolated vertices of G.

Theorem 5

Let A be the adjacency matrix of a simple graph G. Then

$$sp(G) \leq \lambda_1 + (trA^{2\ell} - \lambda_1^{2\ell})^{\frac{1}{2\ell}} \leq 2^{1-\frac{1}{2\ell}} (tr\,A^{2\ell})^{\frac{1}{2\ell}},$$
(13)

where, $\ell \geq 1$.

Equality holds throughout if and only if equality holds in the first inequality, equivalently, if and only if $e=0$ or $G_0 = K_{a,b}$ for some a, b with $e=ab$ and $a+b\leq n$.

Proof: We rewrite $\lambda_1^{2\ell}+\lambda_n^{2\ell} \leq tr\,A^{2\ell}$, as

$$sp(A) \leq \lambda_1 + (trA^{2\ell} - \lambda_1^{2\ell})^{\frac{1}{2\ell}}.$$

Also define

$$f(\lambda) = \lambda + (trA^{2\ell} - \lambda^{2\ell})^{\frac{1}{2\ell}}, \lambda \in [0, (trA^{2\ell})^{\frac{1}{2\ell}}].$$

Then $\lambda^* = (\frac{trA^{2\ell}}{2})^{\frac{1}{2\ell}}$ is a point of absolute maxima of $f(\lambda)$. Now inequality (13) follows on simplification of $f(\lambda^*)$. The equality condition follows as in Theorem 1.5 of [5].

Theorem 6 : Let G be a triangle free graph with n vertices and e edges. Then,

$$sp(A) \geq \frac{1}{\sqrt{2n}}(d_1 + d_2) + \sqrt{d_1} \qquad (14)$$

$$sp(A) \geq \nu + \sqrt{d_1}. \qquad (15)$$

Where, ν is given by (6).

Proof: Since G is a triangle free graph, there is an induced subgraph K_{1,d_1} in G .Thus by interlacing inequality for eigenvalues of a symmetric matrix,

$$\lambda_n \leq \lambda_n(K_{1,d_1}) = -\sqrt{d_1}.$$

Now both these result are clear, in view of Theorem

It is well-known that a graph is bipartite if and only if it has no odd cycle. The above Theorem 5 and Theorem 6 yield the following corollary.

Corollary 7

Examples : We now compare our bounds for the largest eigenvalue of two simple graphs. All numerical eigenvalues are given approximately.

Example 1

Let G be a simple connected graph (as in [6, pp222]):

The upper bounds for this eigenvalue are:

Inequality	(1)	(2)	(3)
Bound value	5.0	3.87	3.44

Table (I)

The lower bounds of inequality (4) and (5) are 2.3094, 1.0 respectively

Example 2

The second graph is the tree, see (2.139, [2, pp285]):

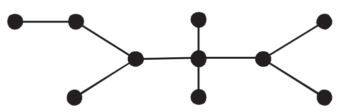

Fig. (2)

The actual value of the largest eigenvalue is 2.307. The upper bounds for this eigenvalue are:

Inequality (1) (2) (3) (10)

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Let G be a bipartite graph with n vertices and e edges. Then

$$\lambda_{\max} \leq 2^{-\frac{1}{2l}} (tr\, A^{2l})^{\frac{1}{2l}}. \qquad (16)$$

Equality conditions are same as for (13) above.

Also,

$$\lambda_{\max} \geq \frac{1}{2}\left(\frac{1}{\sqrt{2n}}(d_1 + d_2) + \sqrt{d_1}\right), \qquad (17)$$

and

$$\lambda_{\max} \geq \frac{1}{2}(\nu + \sqrt{d_1}). \qquad (18)$$

Where, ν is given by (6).

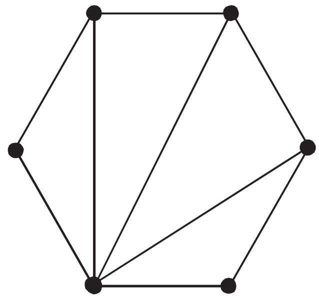

Fig. 1

The actual value of the largest eigenvalue is 3.2227.

Inequality	(1)	(2)	(3)	(10)
Bound value	4.0	4.02	3.23	2.38

Table (II)

Also the lower bounds are:

Inequality	(4)	(5)	(10)	(17)	(18)
Bound value	1.56	1.07	1.66	1.78	1.53

Table (III)

We find that the first upper bound given by (3) does well in example 1 while upper bound in (10) gives a good approximation in example 2. Among the lower bounds in example 2, first lower bound of (10) is the best.

References :

1. Bapat, R.B. (2010). Graphs and Matrices, Hindustan Book Agency, Delhi.
2. Cvetkovic, D.M. , Doob, M. and Suchs, H. (1979). Spectra of graphs, Academic Press, New York.
3. Cvetkovic, D.M. and Rowlinson, P. (1990). The largest eigenvalue of a graph, A Survey, Linear and Multilinear Algebra, 28: 3-33.
4. Golub, G.H. and VanLoan, C.F. (2007). Matrix Computations, Third edition, Hindustan Book Agency.
5. Gregory, D.A., Hershkowitz, D. and Kirkland, S.J. (2001). The spread of the spectrum of a graph, Linear Algebra and its applications 332-334: 23-35.
6. Harary, F., Graph Theory (1988). Narosa Publishing House.
7. Johnson, C.R., Kumar, R. and Wolkowicz, H. (1985). Lower bounds for the spread of a matrix, Linear Algebra and its Applications, 71:161-173.
8. Minc, H. (1988). Nonnegative Matrices, John Wiley and sons.
9. Yu, A. (2010). A new upper bound for the Laplacian spectral radius of a graph, Electronic Journal of Linear Algebra, 20:730-738. Retrieved from http://www.math.technion.ac.i1/iic/ela
10. Wolkowicz, H. and Styan, G.P.H. (1980). Bounds for eigenvalues using Traces, Linear Algebra and its Applications, 29:47

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