A*-ALGEBRAS - SECOND ISOMORPHISM THEOREM

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Abstract: This paper presents definition of A*-algebras, Congruence relation on A*-algebras, Fundamental theorem of homomorphisms of A*-algebras, First Isomorphism theorem on A*-algebras and proof of Second Isomorphism theorem on A*-algebras.

Keywords: A*-algebra, Congruence, Homomorphism, Isomorphism and Sub-A*-algebra.

Definition: An algebra $(A, \wedge, *, (-)^{\sim}, (-)_{\pi}, 1)$ is an A*-algebra if it satisfies:

(i) $a_\pi \vee (a_\pi)^{\sim}=1, (a_\pi)_\pi = a_\pi$ where $a \vee b = (a^{\sim} \wedge b^{\sim})^{\sim}$

(ii) $a_\pi \vee b_\pi = b_\pi \vee a_\pi$

(iii) $(a_\pi \vee b_\pi) \vee c_\pi = a_\pi \vee (b_\pi \vee c_\pi)$

(iv) $(a_\pi \wedge b_\pi) \vee (a_\pi \wedge (b_\pi)^{\sim}) = a_\pi$

(v) $(a \wedge b)_\pi = a_\pi \wedge b_\pi$, $(a \wedge b)^{\#} = a^{\#} \vee b^{\#}$ where $a^{\#} = (a_\pi \vee a^{\sim}{}_\pi)^{\sim}$

(vi) $a^{\sim}{}_\pi = (a_\pi \vee a^{\#})^{\sim}$, $a^{\sim\#} = a^{\#}$

(vii) $(a * b)_\pi = a_\pi$ $(a * b)^{\#} = (a_\pi)^{\sim} \wedge (b^{\sim}{}_\pi)^{\sim}$

(viii) $a = b$ if and only if $a_\pi = b_\pi$, $a^{\#} = b^{\#}$.

We write 0 for $1^{\sim}$, 2 for $0 * 1$.

Example: **3**={0, 1, 2} with the operations defined below is an A*- algebra.

∧	0	1	2
0	0	0	2
1	0	1	2
2	2	2	2

∨	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

*	0	1	2
0	0	2	2
1	1	1	1
2	0	2	2

x	0	1	2
$x^{\sim}$	1	0	2
x_π	0	1	0
$x^{\#}$	0	0	1

Lemma: For any x, y, z in an A*- algebra

(i) $x^{\sim\sim} = x$

(ii) $(x \wedge y)^{\sim}{}_\pi = (x^{\sim} \wedge y)_\pi \vee (x \wedge y^{\sim})_\pi \vee (x^{\sim} \wedge y^{\sim})_\pi$

(iii) $(x \vee y)_\pi = (x^{\sim} \wedge y)_\pi \vee (x \wedge y^{\sim})_\pi \vee (x \wedge y)_\pi$

(iv) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

Lemma: For any x, y in A

(i) $(x * y)^{\sim}{}_\pi = (x_\pi)^{\sim} \wedge (y^{\sim})_\pi$

(ii) $x = x_\pi * (x^{\sim})_\pi{}^{\sim} = (x_\pi) * x^{\#}$

(iii) If $x = e * f$, where $e, f \in B(A)$, $e \wedge f = 0$, then $x_\pi = e$, $x^{\#} = f$.

Theorem: Every A*-algebra $(A, \wedge, *, (-)_\pi, (-)^{\sim}, 1)$ satisfies the following conditions:

For x, y, z in A

(i) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$

(ii) $x \wedge y = y \wedge x$

(iii) $x \wedge x = x$

(iv) $1 \wedge x = x$

(v) $x^{\sim\sim} = x$

(vi) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ where $x \vee y = (x^{\sim} \wedge y^{\sim})^{\sim}$

(vii) $1_\pi = 1$

(viii) $[(x_\pi)^{\sim}]_\pi = (x_\pi)^{\sim}$

(ix) $(x \wedge y)_\pi = x_\pi \wedge y_\pi$

(x) $(x \wedge x^{\sim})_\pi = 0$ where $1^{\sim} = 0$

(xi) $x_\pi \wedge (x_\pi \vee y_\pi) = x_\pi$

(xii) $(x \wedge y)^{\sim}{}_\pi = (x \wedge y^{\sim})_\pi \vee (x^{\sim} \wedge y)_\pi \vee (x^{\sim} \wedge y^{\sim})_\pi$

(xiii) $(x_\pi)_\pi = x_\pi$

(xiv) $(x * y)_\pi = x_\pi$

(xv) $(x * y)^{\sim}{}_\pi = (x_\pi)^{\sim} \wedge (y^{\sim})_\pi$

(xvi) $x = x_\pi * (x^{\sim}{}_\pi)^{\sim}$

Theorem: An algebra $(A, \wedge, *, (-)^{\sim}, (-)_\pi, 1)$ satisfying axioms of the above theorem is an A*-algebra.

Definition: Let $(A, \wedge, *, (-)^{\sim}, (-)_\pi, 1)$ be an A*-algebra and $A_1 \subseteq A$, A_1 is called a sub A*-algebra of A if A_1 is closed under $\wedge, *, (-)^{\sim}, (-)_\pi, 0, 1$.

Definition: An A*-algebra of sets (with universal set X) is a subset of T_X, closed under $\wedge, \vee, (-)^{\sim}, (-)_\pi, *$.

Definition: Let $(A_1, \wedge, \vee, (-)^{\sim}, (-)_\pi, *, 1)$ and $(A_2, \wedge, \vee, (-)^{\sim}, (-)_\pi, *, 1)$ be A*- algebras. A mapping $f : A_1 \to A_2$ is called an A*-homomorphism if

(i) $f(a \wedge b) = f(a) \wedge f(b)$ (ii) $f(a * b) = f(a) * f(b)$

(iii) $f(a_\pi) = (f(a))_\pi$ (iv) $f(a^{\sim}) = (f(a))^{\sim}$

(v) $f(1) = 1$ and (vi) $f(0) = 0$.

If in addition f is bijective, then f is called an A*-isomorphism, and A_1, A_2 are said to be isomorphic, denote in symbols $A_1 \cong A_2$.

Definition: A congruence relation $\emptyset$ on an A*-algebra is an equivalence relation on A satisfying

(i) $a \emptyset b \Rightarrow a_\pi \emptyset b_\pi$, $a^{\#} \emptyset b^{\#}$, $a^{\sim} \emptyset b^{\sim}$

(ii) $a \emptyset b, c \emptyset d \Rightarrow (a * c) \emptyset (b * d)$, $(a \wedge c) \emptyset (b \wedge d)$.

Note: The above definition is equivalent to

(i) $a \emptyset b \Rightarrow a^{\sim} \emptyset b^{\sim}$

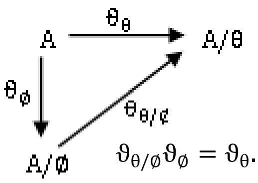
(ii) $a \emptyset b, c \emptyset d \Rightarrow (a \wedge c) \emptyset (b \wedge d)$

(iii) $a \emptyset b, c \emptyset d \Rightarrow (a * c) \emptyset (b * d)$.

Theorem: Fundamental theorem of homomorphisms of A*-algebras:
Let A and B be A*-algebras, f a homomorphism of A into B. Then $\emptyset = f^{-1}f$ is a congruence on A and f(A) is a sub A*-algebra of B. Moreover, we have a unique homomorphism $\bar{f}$ of A/∅ into B such that $f = \bar{f}\vartheta$, where ϑ is the homomorphism $a \mapsto \bar{a}$ where $\bar{a} = \emptyset(a)$ of A into A/∅. The homomorphism $\bar{f}$ is injective and ϑ is surjective.

Theorem: First isomorphism theorem on A*-algebras:
Let ∅ be a congruence on an A*-algebra A, A_1 a sub A*-algebra of A. Let A'_1 be the union of the ∅ - equivalence classes that meet A_1. Then A'_1 is a subalgebra of A containing A_1, $\emptyset_1 = \emptyset \cap (A_1 \times A_1)$ and $\emptyset'_1 = \emptyset \cap (A'_1 \times A'_1)$ are congruences on A_1 and A'_1 respectively, and $\bar{a}_{1\emptyset_1} \mapsto \bar{a}_{1\emptyset'_1}$ is an isomorphism of $A_1/\emptyset_1$ onto $A'_1/\emptyset'_1$.

Theorem: Suppose A is an A*-algebra and ∅ is a congruence relation on A. Suppose θ is another congruence on A. Define $\vartheta_{\theta/\emptyset} : A/\emptyset \to A/\theta$ as $\bar{a}_\emptyset \mapsto \bar{a}_\theta$. Then $\vartheta_{\theta/\emptyset}$ is a map if and only if $\emptyset \subseteq \theta$. So, in this case $\vartheta_{\theta/\emptyset}: \bar{a}_\emptyset \mapsto \bar{a}_\theta$ is a unique map such that



is commutative, i.e., $\vartheta_{\theta/\emptyset}\vartheta_\emptyset = \vartheta_\theta$.

Note: $\theta/\emptyset = \{ \bar{a}_\emptyset \in A/\emptyset \mid \bar{a}_\theta = \bar{0}_\theta\}$. This is also called kernel of $\vartheta_{\theta/\emptyset}$.

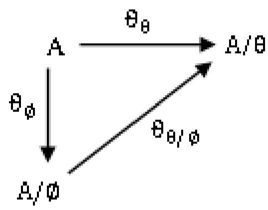
Note: Define $\theta/\emptyset = \{(\bar{a}_\emptyset, \bar{b}_\emptyset) \mid \bar{a}_\emptyset, \bar{b}_\emptyset \in A/\emptyset$ and $\bar{a}_\theta = \bar{b}_\theta\}$. θ/∅ is also called kernel of $\vartheta_{\theta/\emptyset}$.

Theorem: θ/∅ is a congruence relation on A/∅.

Theorem: Suppose θ_1, θ_2 are two congruences on an A*-algebra A such that $\theta_1 \supset \emptyset$, $\theta_2 \supset \emptyset$. Then $\theta_1 \supset \theta_2$ if and only if $\theta_1/\emptyset \supset \theta_2/\emptyset$. In particular, $\theta_1/\emptyset = \theta_2/\emptyset$ implies $\theta_1 = \theta_2$.

Theorem: Any congruence $\bar{\theta}$ on A/∅ has the form θ/∅, where θ is a congruence relation on the A*-algebra A such that $\theta \supset \emptyset$.

Theorem: Let A be an A*-algebra, ∅ be a congruence on A, A/∅ the corresponding quotient algebra. Let θ be a congruence on A such that θ contains ∅. Then there exist a unique homomorphism $\vartheta_{\theta/\emptyset} : A/\emptyset \to A/\theta$ such that



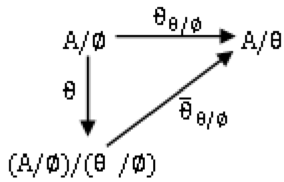
is commutative, i.e.,

$$\vartheta_{\theta/\emptyset}\vartheta_\emptyset = \vartheta_\theta.$$

and if θ/∅ denotes the kernel of $\vartheta_{\theta/\emptyset}$, then θ/∅ is a congruence on A/∅. The map $\theta \mapsto \theta/\emptyset$ is a bijective map of the set of congruences on A/∅. Moreover, $\theta_1 \supset \theta_2$ are two congruences on A containing ∅ if and only if $\theta_1/\emptyset \supset \theta_2/\emptyset$.

Theorem: Second isomorphism theorem on A*-algebras:
Let θ and ∅ be congruences on the A*- algebra A such that $\theta \supset \emptyset$ and let θ/∅ be the corresponding congruence of A/∅. Then $(\overline{\bar{a}_\emptyset})_{\theta/\emptyset} \mapsto \bar{a}_\theta$, $a \in A$ is an isomorphism of (A/∅)/(θ /∅) onto A/θ.

Proof: The homomorphism $\vartheta_{\theta/\emptyset} : A/\emptyset \to A/\theta$ is an epimorphism and has kernel θ/∅. Therefore θ/∅ is an equivalence relation on A/∅.
By Fundamental theorem of A*-algebras there exist a unique morphism $\bar{\vartheta}_{\theta/\emptyset}$ of (A/∅)/(θ /∅) into A/θ such that the diagram



commutative, i.e.,

$$\bar{\vartheta}_{\theta/\emptyset}\vartheta = \vartheta_{\theta/\emptyset}$$

and $\bar{\vartheta}_{\theta/\emptyset}$ is injective and ϑ is surjective. Since, ϑ, $\vartheta_{\theta/\emptyset}$ are epimorphisms and $\bar{\vartheta}_{\theta/\emptyset}\vartheta = \vartheta_{\theta/\emptyset}$, $\bar{\vartheta}_{\theta/\emptyset}$ is surjective. Therefore $\bar{\vartheta}_{\theta/\emptyset}$ is an isomorphism.

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