A COMPLEX NUMBER TYPE REPRESENTATION FOR HARMONIC ANALYSIS IN HIGHER DIMENSIONAL SPACES

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Abstract : We propose a complex type number system for the space of real sequences. We show that the number system is a commutative division algebra forming a field. It is an extension of the complex number system and its exponential and trigonometric properties. Based on this new complex type system, we construct Cauchy-Riemann type conditions for complex type valued holomorphic functions of complex type variable. Hence, we construct a basis for the space of harmonic functions defined on the three dimensional space.

Key Words : Cauchy-Riemann conditions, Euler's formula, Harmonic polynomials.
Hyper-complex numbers

Introduction: The history of searching for higher dimensional numbers that behave like real numbers is very long. In this connection one can refer [1] and the references therein. The only other number system known to us, besides real, is the complex field. Besides its purely theoretical aspect, complex numbers have been proved useful in solving numerous mathematical modeling problems in almost all the fields of science.

Many complex-like extensions for higher dimensional spaces have been proposed in the past. Quaternion [2], Octonion [3], hypercomplex numbers [4] and Clifford Algebra [5] are examples of early attempts in this direction. Recently, Fleury et al. [6][7] have defined a multi-complex number system which is an n-dimensional R-algebra generated by a fundamental unit e fulfilling $e^n = -1$ [6].

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All these extensions were defined on finite dimensional spaces. In order to formulate these extended complex spaces, some relaxation of some field properties had to be made. For example, the commutative property is relaxed in Quaternion and the associative property is relaxed in Octonion. Fleury et al. relax the invertiblity of all non-zero multi-complex numbers. That is, in their formulation, there are non-zero multi-complex numbers without inverse.[6]

In this paper, we propose an extended complex type number system which retains all the axioms of a field. The complex type system is based on a new class of multicomplex algebra defined in [11] in an infinite dimensional space of sequences of real numbers. The sequences are represented by their generating functions giving closed forms for the complex type operations.

The motivation of the definition of complex type system comes from an attempt for a generalization of the fact that the complex valued analytic functions of the two dimensional Cartesian coordinate variables x and y constitute a pair of real valued harmonic functions. Our complex type representation gives a canonical extension of this property to functions of higher dimensional coordinate variables. Besides, we see that other basic properties of complex numbers such as trigonometric functions representation, De Moivre's formula and Euler's formula for exponential representations are also satisfied by the complex type field with some generalizations.

Our construction of complex type numbers can be readily extended to satisfy the harmonic function property for functions in any finite dimensional spaces. In this paper, however, we restrict our discussion to functions in three dimensional variables only. In this context, we see that our complex type representation is a natural extension of the classical complex space in the sense that the complex space is included in our complex type space and that the fundamental algebraic, trigonometric, exponential and holomorphic properties of functions are carried to the extended complex type space.

This paper is organized as follows: In Section 2, the definition of complex type number system and its basic properties are given. In Section 3, the extended trigonometric and exponential representation and their identities are established. In Section 4, Analytical functions and its properties are described. In Section 5, a basis for the space of harmonic functions in three dimensional space is derived. In Section 6, a definition for further extension of the complex type space is provided and finally, a conclusion is drawn in Section 7.

2.**Definition And Basic Properties**: Let t be a symbolic variable and (t) be the space of infinitely differentiable functions with real coefficients. We define a complex type space C(t) as the set of pairs of functions $(a(t), b(t)) \in \mathrm{P}(t)^2$ with the usual addition along with a complex type multiplication. When there is no ambiguity, we omit the argument variable t for the functions for notational brevity.

Definition 2.1. The complex type space

$C(t) = \{(a_1(t), a_2(t)) \mid a_1(t), a_2(t) \in P(t)\}$ is defined with operations

1. $a + b = (a_1 + b_1, a_2 + b_2)$,
2. $ab = (a_1 b_1 - (1+t^2) a_2 b_2, a_1 b_2 + a_2 b_1)$,

where $a = (a_1, a_2), b = (b_1, b_2)$ and $a_i, b_i \in P(t), i = 1,2$.

Although we define the space consisting of functions, we are mainly interested in the coefficients of the powers of t in the infinite series representation of the functions. In this context, the functions are regarded as generating functions of their coefficients. We denote the sequence of coefficients of a function $a(t)$ by a itself so that when $a = \{a_n\}_{n=0}^{\infty}$, we have $a(t) = \sum_{n=0}^{\infty} a_n t^n$. We refer the pairs of functions or the pairs of sequences as the 'complex type numbers' and denote $(a(t); b(t))$ for function form and $(a; b)$ for sequence form. We use both notations interchangeably as they point to the same number.

Alternatively, the elements of the space ℂ(t) can also be expressed in the complex type form $a(t) + jb(t)$, where the 'imaginary' symbol j satisfies $j^2 = -(1+t^2)$ owing to the usual properties of field established in the following proposition.

Proposition 2.2. The space of complex type numbers ℂ(t) forms a field under the defined operations. The elements of ℂ(t) can be expressed in the complex type form a+jb, where $j^2 = -(1+t^2)$.

Proof. Closed, associative, commutative and distributive properties of the operations are immediate by algebraic manipulations. The zero element is $(0(t); 0(t))$, where o(t) = o and the unit element is $(1(t); 0(t))$, where $1(t) = 1$. The inverse of a non-zero complex type number $f = (a; b)$ is

$$f^{-1} = \frac{(a; -b)}{a^2 + (1+t^2) b^2}.$$

The second elementary basis element $j = (0(t); 1(t))$ satisfies a coupled relation $j^2 = -(1+t^2)(1(t); 0(t)) = -(1+t^2)$.

It should be pointed out that the fundamental imaginary unit j is a function of the symbolic variable t. In the previous extended number formulations, the fundemental units e satisfy conditions of the form $e^n = \pm 1$ for some power n. This is one of the main differences between the present complex type formulation and the previous other formulations.

Definition 2.3. For a complex type number $f = a + jb$, we call a and b, the real type and imaginary type parts of f respectively. Further, we define the complex type conjugate as $f^* = a - jb$ and the complex type modulus as $|f| = +\sqrt{a^2 + (1+t^2) b^2}$.

By this definition, we see that the system of complex type numbers has canonical extensions of many properties of its complex counterparts.

Note that the complex type modulus is not a real number. Rather, it is a generating function consisting of sequences of real numbers as coefficients.

Proposition 2.4. For the complex type numbers $a, b \in X(t)$, we have

i. $|a^*| = |a|$, ii. $|ab| = |a||b|$,

iii. $aa^* = |a|^2$, iv. $a^{-1} = \dfrac{a^*}{|a|^2}, a \neq 0$,

v. ab = o if and only if a = o or b = o,

vi. | a | = o if and only if a = o.

Proof. Proof of i. to v. are direct by algebraic manipulations.

To prove vi., it is convenient to define generating functions $a_n(t)$ for a sequence $a = \{a_n\}_{n=0}^{\infty}$ as for each n, $a_n(t) = a_n + a_{n+1} t + \cdots$, so that we have $a_n(t) = a_n + t a_{n+1}(t), n = 0,1,2,\cdots$.

For $w = a(t) + jb(t)$, we prove inductively that $a_n = b_n = 0$ for all $n = 0,1,2,\cdots$.

$$\begin{aligned} |a|^2 &= a_0(t)^2 + (1+t^2) b_0(t)^2 \\ &= (a_0 + t a_1(t))^2 + (1+t^2)(b_0 + t b_1(t))^2 \\ &= a_0^2 + b_0^2 + 2t(a_0 a_1(t) + b_0 b_1(t)) \\ &\quad + t^2 (a_1(t)^2 + b_1(t)^2 + b_0(t)^2) = 0. \end{aligned}$$

Equating the constant terms, we get $a_0^2 + b_0^2 = 0$. Hence, $a_0 = b_0 = 0$.

Suppose inductively, that $a_i = b_i = 0$ for $i = 0,1,2,\cdots n-1$. Then, | a | = o reduces to

$$|a|^2 = (t^n a_n(t))^2 + (1+t^2)(t^n b_n(t))^2 = 0$$

which gives

$$\begin{aligned} &a_n^2 + b_n^2 + 2t(a_n a_{n+1}(t) + b_n b_{n+1}(t)) \\ &\quad + t^2 (a_{n+1}(t)^2 + a_{n+1}(t)^2) = 0. \end{aligned}$$

Thus we have $a_n^2 + b_n^2 = 0$ and hence, $a_n = b_n = 0$.

We see that our complex type space ℂ(t) is an extension of the complex space ℂ in the sense that ℂ is included in ℂ(t). Specifically, the subspace ℂ(o) of ℂ(t) is the complex space consisting the constant coefficients of the real and imaginary type parts of complex type numbers.

One can easily see that complex type multiplication of a complex type number with a real number reduces to the usual scalar multiplication of the linear space of real sequences. The linear space is therefore infinite dimensional.

When the real and imaginary type parts of a complex type number are finite sequences of size m and n respectively, their generating functions are polynomials in t of degrees m - 1 and n - 1 respectively. We call this a complex type number of dimensions $[m,n]$. The total dimension of the number is defined to be m + n.

Remark 2.5. We note in passing that the multiplication of two complex type numbers of dimensions $[m,n]$ does not result in a number of the same dimension. More generally, if Z_1, Z_2 are complex type numbers of dimensions $[m_1,n_1],[m_2,n_2]$ respectively, then Z_1Z_2 is of dimensions

$$[\max(m_1+m_2-1, n_1+n_2+1), \max(m_1+n_2-1, n_1+m_2-1)].$$

Therefore, the multiplication is not restricted to a fixed finite dimension. This is another main difference between our formulation and the previous formulations. This property allows us to construct a complete basis for the solution space of the Laplace's equation(see Section 5).

Exponential And Trigonometric Representations: One may pose an immediate question whether the complex type numbers can be expressed in an Euler type form with exponentials. The following results show that the trigonometric and exponential functions representations and their properties in the classical complex space have canonical extensions in the complex type space.

First, we consider functions of a complex type variable. If $w(t)$ is a complex type variable and $F(.)$ is an infinitely differentiable univariate function, we may write $F(w)$ as an infinite series $F(w)=\sum_{n=0}^{\infty}\frac{F^{(n)}(0)}{n!}w^n(t)$. Since ⍰(t) is a field, there is no restriction in ascertaining that $F(w)$ is a complex type valued function. We use this fact to establish the trigonometric and exponential representations of a complex type number.

heorem 3.1. The complex type numbers $w \in \mathrm{X}(t)$ can be expressed in the exponential and trigonometric type representations $w = a + jb = \rho e^{j\theta} = \rho(\cos_t \theta + j\sin_t \theta)$, where $\theta = \theta(t), \rho = \rho(t) = |w| \in \mathrm{P}(t)$ and the trigonometric type functions are defined as

$$\cos_t \theta := \cos(\sqrt{1+t^2}\theta) = \frac{a}{\rho}, \qquad (1)$$

and $$\sin_t \theta := \frac{\sin(\sqrt{1+t^2}\theta)}{\sqrt{1+t^2}} = \frac{b}{\rho}. \qquad (2)$$

Further, the trigonometric type functions satisfy the De Moivre's formula:

For an integer n,

$(\cos_t \theta + j\sin_t \theta)^n = \cos_t n\theta + j\sin_t n\theta$.

Proof. By the last remark of this section, $\ln(w)$ is a complex type valued function and hence we may write $\ln(w(t)) = \lambda(t) + j\theta(t)$, where we consider the principal branch of logarithm, and hence we have $w = e^{\lambda + j\theta} = \rho e^{j\theta}$ with $\rho = e^{\lambda}$.

Now,

$$e^{j\theta} = e^{i\sqrt{1+t^2}\theta} = \cos\sqrt{1+t^2}\theta + i\sin\sqrt{1+t^2}\theta$$
$$= \cos_t \theta + j\sin_t \theta,$$

where the trigonometric type functions are defined by (1) and (2). Again, by Proposition 4, we have $|e^{j\theta}| = |e^{i\sqrt{1+t^2}\theta}| = 1$, and hence $\rho = |w|$. Here, the relations between the exponential and trigonometric type functions are establish regarding them as infinite series.

The De Moivre's type formula is then an immediate consequence by induction. ■

Here, we have identified $\theta(t)$ from the concepts of infinite series representations of the functions involved. Therefore, it is not known whether the real coefficients of $\theta(t)$ have any geometric or trigonometric meaning. It is, however, observed that the first coefficient $\theta(0)$ is the classical angular argument of the complex number $w(0)$.

The trigonometric type functions defined in Theorem 6 satisfy some extended versions of many trigonometric identities. For example,

Theorem 3.2. The trigonometric type functions satisfy the following identities:

1. $\cos_t^2 \theta + (1+t^2)\sin_t^2 \theta = 1,$
2. $\cos_t(\theta_1 \pm \theta_2)$
$= \cos_t \theta_1 \cos_t \theta_2 \mp (1+t^2)\sin_t \theta_1 \sin_t \theta_2,$
3. $\sin_t(\theta_1 \pm \theta_2)$
$= \sin_t \theta_1 \cos_t \theta_2 \pm \cos_t \theta_1 \sin_t \theta_2,$

4. $$\cos_t 2\theta = \cos_t^2 \theta - (1+t^2)\sin_t^2 \theta = 1 - 2(1+t^2)\sin_t^2 \theta = 2\cos_t^2 \theta - 1,$$

5. $\sin_t 2\theta = 2\sin_t \theta \cos_t \theta$.

We omit here for brevity, the proof of Theorem 3.2 and the definition of other trigonometric type functions and their identities as they can be obtained by direct manipulations of their classical versions.

Analytical Functions In Three Dimentional Space: In this section, we turn our attention to the three dimensional space with Cartesian coordinates $(x, y, z) \in \mathrm{P}^3$. Our interest here is in the functions g(x,y,z) of the coordinate variables. For a sequence of such functions $\{g_n(x,y,z)\}_{n=0}^{\infty}$, we denote its generating function as g(x,y,z)(t) and we use the same notation for a complex type valued function formed by two such sequences.

We define the complex type number $w(t) = x + yt + jz$. Concerning functions of the variables x,y and z , we make the following definition.

Definition 4.1. A complex type valued function $f(x,y,z)(t) = f_1(x,y,z)(t) + jf_2(x,y,z)(t)$ is called analytic if there exists a differentiable function F(.) of one variable such that f(t) = F(w(t)), where $w(t) = x + yt + jz$.

Theorem 4.2. (Cauchy-Riemann type Formula) The real and imaginary type parts of a complex type valued analytic function $f(x,y,z)(t) = f_1(x,y,z)(t) + jf_2(x,y,z)(t)$ satisfy the Cauchy-Riemann type formula

$$t\frac{\partial f_1}{\partial x} = \frac{\partial f_1}{\partial y}, \quad (3)$$

$$t\frac{\partial f_2}{\partial x} = \frac{\partial f_2}{\partial y}, \quad (4)$$

$$\frac{\partial f_1}{\partial x} = \frac{\partial f_2}{\partial z}, \text{ and} \quad (5)$$

$$\frac{\partial f_1}{\partial z} = -(1+t^2)\frac{\partial f_2}{\partial x}. \quad (6)$$

Proof. Since f is analytic, we have f = F(w(t)) for some univariate differentiable function F(.). Differentiating f with respect to x and y using the chain rule of differentiation, and eliminating $F'(w)$, we get (3) and (4) by equating the real and imaginary type parts. Again, differentiating f with respect to x and z and eliminating $F'(w)$, we get (5) and (6).

The Cauchy-Riemann type relations form a sequence of relations among the coefficients of the real and imaginary type functions. If $f_1 = \{f_{i,n}\}_{n=0}^{\infty}$ and $f_2 = \{f_{2,n}\}_{n=0}^{\infty}$, then the Cauchy-Riemann type relations means the following:

$$\frac{\partial f_{1,n}}{\partial x} = \frac{\partial f_{1,n+1}}{\partial y} = \frac{\partial f_{2,n}}{\partial z},$$

$$\frac{\partial f_{2,n}}{\partial x} = \frac{\partial f_{2,n+1}}{\partial y},$$

$$\frac{\partial f_{2,n}}{\partial x} + \frac{\partial f_{2,n+2}}{\partial x} + \frac{\partial f_{1,n}}{\partial z} = 0.$$

It is, therefore, possible to construct a complex type valued analytic function upto a constant from a given infinitely differentiable function.

A function u(x,y,z) is called harmonic if it satisfies the Laplace's equation $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$. We call a generating function g(x,y,z)(t) harmonic when all of its coefficients (the sequence terms) are harmonic.

Theorem 4.3. The real and imaginary type parts of an analytic function are harmonic.

Proof. For an analytic function $f(x,y,z)(t) = f_1(x,y,z)(t) + jf_2(x,y,z)(t)$ we have the Cauchy-Reimann formula (3)-(6).

Differentiating (5) and (6) with respect to x and z respectively, we have $\frac{\partial^2 f_1}{\partial x^2} = \frac{\partial^2 f_2}{\partial x \partial y}$ and $\frac{\partial^2 f_1}{\partial z^2} = -(1+t^2)\frac{\partial^2 f_2}{\partial x \partial z}$. Eliminating the mixed derivative from these, we obtain $\frac{\partial^2 f_1}{\partial z^2} = -(1+t^2)\frac{\partial^2 f_2}{\partial x^2}$. Again, differentiating (3) with respect to x and y, we have $t\frac{\partial^2 f_1}{\partial x^2} = \frac{\partial^2 f_1}{\partial x \partial y}$ and $t\frac{\partial^2 f_1}{\partial x \partial y} = \frac{\partial^2 f_1}{\partial y^2}$ respectively. Thus, we obtain $t\frac{\partial^2 f_1}{\partial y^2} = \frac{\partial^2 f_1}{\partial x^2}$. Now, eliminating t^2 from the two equations obtained, we see that $\frac{\partial^2 f_1}{\partial x^2} + \frac{\partial^2 f_1}{\partial y^2} + \frac{\partial^2 f_1}{\partial z^2} = 0$. A similar approach using (4),(5) and (6) yields the result for f_2.

A Basis For Harmonic Functions In Three Dimensions:. We now use the results on analytic functions to construct a basis for the harmonic function space in three dimensions. It is known that the space of harmonic functions of a mode n has dimensions 2n+1. The following theorem establishes a basis for the harmonic space.

Theorem 5.1. The set of real valued functions of the Cartesian coordinate variables x,y and z defined by the coefficients of the complex type valued function $w^n(t) = (x + yt + jz)^n$, $n \in \mathrm{N}_0 = \mathrm{N} \bigcup \{0\}$ form a basis for the space of harmonic functions.

Proof. It is enough to prove that $w^n(t) = (x + yt + jz)^n$ has 2n + 1 independent coefficient functions. By the trinomial expansions, we have

$$w^n(t) = \sum_{k+l+m=n} \binom{n}{k,l} x^k (yt)^l (jz)^m$$

$$= \sum_{\substack{k+l+m=n \\ m\ even}} \binom{n}{k,l} x^k y^l z^m (1+t^2)^{m/2} t^l$$

$$+ j \sum_{\substack{k+l+m=n \\ m\ odd}} \binom{n}{k,l} x^k y^l z^m (1+t^2)^{(m-1)/2} t^l$$

$$= \sum_{\substack{k+l+m=n \\ m\ even}} \sum_{r=0}^{m/2} \binom{n}{k,l} \binom{m/2}{r} x^k y^l z^m t^{l+2r}$$

$$+ j \sum_{\substack{k+l+m=n \\ m\ odd}} \sum_{r=0}^{(m-1)/2} \binom{M}{r} \binom{n}{k,l} x^k y^l z^m t^{l+2r}$$

where $M = (m-1)/2$.

The first term is a polynomial in t of degree n and the second term consists of polynomial of degree n - 1. Thus, the number of coefficients are n + 1 and n respectively. Each coefficient is homogeneous polynomials in the variables x,y and z in which all the terms have different powers with total power n. Hence, they are independent.

We recall that this basis for the space of harmonic functions had been constructed earlier from other concepts (see for example, [8],[9] and [11]). Miles and William [8] give an explicit form for the harmonic bases for function in k variables. Ketchum [9] expresses the basis in an integral form which requires some additional work to construct the explicit form of the basis harmonic polynomials. Recently, the present author [10] constructed a similar form for spherical harmonics by solving the Laplace-Beltrami operator on the sphere for a non-polar spherical coordinate system.

We see that the proposed complex type representation provides a unified platform to extend the theories of complex numbers and theories for harmonic functions for higher dimensions.

Further Extensions : In this section, we state the general definition of extended complex type numbers which are applicable to holomorphic theories for functions defined in higher dimensional coordinate spaces.

Definition 6.1 (Extended Complex type Numbers). Let $\mathbf{t} = (t_1, t_2, \cdots, t_{n-2})$ be a vector of symbols of size n – 2. For a tuple $(\alpha) = (i_1, i_2, \cdots, i_{n-2})$ of non-negative integers, denote $\mathbf{t}^{(\alpha)} = t_1^{i_1} t_1^{i_2} \cdots t_{n-2}^{i_{n-2}}$ and $f_{(\alpha)} = f_{i_1, i_2, \cdots, i_{n-2}}$. Let $P(\mathbf{t})$ be the space of all functions expressible in infinite series form $f(\mathbf{t}) = \sum_{(\alpha)} f_{(\alpha)} \mathbf{t}^{(\alpha)}$, where the summation is over all possible $(n-2)$-tuples (α).

The extended complex type space □(t) is defined as the set $C(\mathbf{t}) = \{a = (a_1(\mathbf{t}), a_2(\mathbf{t})) \mid a_1(\mathbf{t}), a_2(\mathbf{t}) \in R(\mathbf{t})\}$ with operations defined by

1. $a + b = (a_1 + b_1, a_2 + b_2)$,
2. $ab = (a_1 a_2 - (1+\mathbf{t}^2) b_1 b_2, a_1 b_2 + a_2 b_1)$,

where $a_i, b_i \in \mathrm{P}(\mathbf{t}), i = 1,2$ and $\mathbf{t}^2 = t_1^2 + t_2^2 + \cdots + t_{n-2}^2$.

One can easily see that all the above definitions and results in this paper hold for this extended complex type space with similar extensions. The extended complex type numbers can be expressed as $f = a + jb$, where $j^2 = -(1+\mathbf{t}^2)$.

The extended complex type space is a generalization of the complex space in the sense that all the results for this space reduce to their classical complex counterparts when the symbolic variable t = 0=(0,0,0,...,0).

Conclusion : A complex type space for real sequences is proposed. The complex type space is a canonical extension of the classical complex space in the sense that the fundamental trigonometric, algebraic and exponential properties are carried to the proposed space with some extensions. The holomorphic properties of functions defined on the higher dimensional coordinate variables also have generalized extension.

Further properties and analysis of the classical complex space may be generalized with appropriate modifications

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