A NEW CLASS OF MULTICOMPLEX ALGEBRA WITH APPLICATIONS

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Abstract : Hyper complex and multicomplex number systems are extensions of the concept of the fields of real and complex numbers. There have been extensive study and attempts to construct, analyze and classify these systems for over a century. The well known strategy to construct hyper complex systems is the Cayley-Dickson process. In this work, we extend the concept of Cayley-Dickson process to construct some new classes of multicomplex systems. We give examples for the new classes and their applications in constructing general solutions of partial differential equations. We also demonstrate that some infinite dimensional systems with field properties are included in these new classes.

Keywords: Cayley-Dickson, Division algebra, Differential equation, Multicomplex

Introduction : Hyper complex number systems are the outcomes of attempts to extend the one dimensional real and the two dimensional complex number systems. From the middle of the 19th century onwards, since the discovery of Quaternion by R.W. Hamilton[1], many hyper complex systems have been constructed, analyzed and their applications have been proposed. For early reviews on this subject one can consult [2].

On the construction of hyper complex systems, Cayley-Dickson extension from a base hyper complex system to double its dimension is well known. By this process, the Complex, the Quaternion and the Octonion have been constructed to form division algebras[3].

In the early history of hyper complex numbers, we see that researchers were attempting to retain the entire field properties which are intact with the real and complex systems. For this some compromises on some of the field properties had to be made. The Quaternion is not commutative and the Octonion[4] is not even associative. The next step leads to Sedonion which fails to be even division algebra[5].

Recent developments in this direction indicate that modern researchers prefer to keep the commutative and associative properties in exchange for the division algebra [6] [7]. Applications in areas of sciences involving mathematics have been shown to be promising in this compromise [8]. Based on this new line of view, several commutative hyper complex systems recently known as multicomplex systems have been constructed, analyzed and classified to some extent.

In this paper, we construct further new classes of multicomplex systems and show that they have some applications in finding solutions to partial differential equations. First, we give some classification schemes for hyper complex systems and show that the existing multicomplex systems come under these classes within a subclass of commutative algebra. Next, we propose a new class of multicomplex systems and give further examples for this class. We also give some applications for the new class examples.

The paper is organized as follows: In Section 2 we give an account of the existing hyper complex systems and multicomplex systems with some known classifications along with some clarifications. In Section 3, we propose a new extension of the Cayley-Dickson process and construct new classes of multicomplex system. In Section 4 we list some examples for this new class with possible applications. In Section 5, we draw some conclusions.

Preliminaries And Notations: A hyper complex number system is an algebra with unity. It is a division algebra if each of its non-zero elements has a multiplicative inverse. Otherwise, there will be non-zero elements called zero divisors a, b such that ab = 0. A commutative hyper complex system has its multiplication commutative and is called multicomplex system.

A hyper complex number system is viewed as a vector space over a field F (usually R or C) and its elements are vectors expressed in terms of a basis. The elements of a basis are the unit element denoted by 1 and the imaginary symbols $i_k, k \in I$, where I is some countable indexed set. A hyper complex number is expressed, with $i_0 := 1$, as

$$w = a_0 1 + a_1 i_1 + a_2 i_2 + a_3 i_3 + \cdots = \sum_{k \in I} a_k i_k,$$ where

$a_k \in F$ and the summation ranges over all symbols. It is said to have finite rank n if, as a vector space, it has dimension n and so has $n-1$ imaginary symbols.

Early examples of commutative and non-commutative hyper complex systems include the real, complex, quaternion and the octonion. The first two are fields while the others are not. In fact, the quarternion is non-commutative division algebra while the octonion is non-associative division algebra.

Other extensions include Clifford algebras which are also non-commutative algebras excepting complex.

Analysis of hyper complex systems of finite ranks more than 2 shows that there is a tradeoff between the commutativity and invertibility. Recent developments indicate that it is more convenient and useful to include commutative property at the expense of global inverse.

Cayley-Dicksion Extension

It is possible to extend a hyper complex system to double its rank by the Cayley-Dickson(C-D) process[3]. If X is a hyper complex system of rank n with symbols $i_k, 1 \le k \le n-1$, an extended system Y of rank 2n is constructed by elements of the form $a + jb$, where $a, b \in X$, and the new symbol j satisfies

$$j^2 = -1. \quad (1)$$

Other new symbols of the form ji_k arising from multiplication are chosen such that $ji_k = -i_k j$ to ensure that Y is a division algebra, compromising commutativity for ranks larger than 2. The multiplication in the C-D process is then given as $(a + jb)(c + jd) = (ac + d^* b) + j(ad + c^* b),$ where the superscript * indicates involution (conjugate) satisfying the properties

$$(a^*)^* = a \text{ and } (ab)^* = (b^* a^*). \quad (2)$$

The C-D extension of the real system R of rank 1 is the complex system C with rank 2, again extended to the quaternion Q of rank 4, then to the octonion O of rank 8. The next extension sedonion fails to be a division algebra ending the endeavor of constructing division algebras through C-D process.

On the other hand if one allows commutative property for multiplication, non-division algebras can be constructed without ends with ranks 2^n at the nth extension. In this case, each of the hyper imaginary symbols satisfy the property

$$i_k^{\ 2} = \pm 1. \quad (3)$$

Remark 2.1: More classes of hyper complex systems are constructed by allowing the square symbols to assume zero. That is, $i_k^{\ 2} = 0.$ This is known as the Dual or Study complex system. In this paper, we exclude this as it requires separate treatment.

Further classes of extended multicomplex systems have been constructed by generalizing the square condition to the nth power[6]. i.e. $\alpha^n = -1$. This leads to the following definition.

Definition 2.2. A hyper complex system is called cyclic if its basis units satisfy, $i_k = \alpha^k, k = 0,1,2,\cdots$ with $i_0 = \alpha^0 = 1$, for some imaginary symbol α which is called the fundamental imaginary unit or the imaginary generator.

Clearly, a cyclic hyper complex system is commutative and hence is a multicomplex system. With appropriate normalization, a cyclic multicomplex system of finite rank n has the property that $\alpha^n = \pm 1$.

We denote a generic hyper complex system by HC, a system with rank n by $HC(n)$ and a finite rank cyclic multicomplex system by $MC(n, \varepsilon)$, where $\alpha^n = \varepsilon = \pm 1$. $MC(\infty)$ will denote a cyclic multicomplex system of infinite rank. We also denote the set of symbols as BHC(.) or BMC(.). Note that in the hyper complex systems, excluding the Octonion and Sedonion, the set of symbols $G = BHC(.) \cup -BHC(.) := G^+ \cup G^-$, where $-BHC(.)$ consists of the additive inverses (negatives) of $BHC(.)$, form a group under the multiplication of $HC(.)$. In fact, one constructs a hyper complex system over a field F from a group G of symbols by the set of elements $a = \sum_{i_l \in G} a_l i_l$ with multiplication of two elements defined by $ab = \left(\sum_{i_k \in G} a_k i_k \right) \left(\sum_{i_l \in G} b_l i_l \right) = \left(\sum_{i_m \in G} c_m i_m \right)$. If G is commutative then so is HC(.). We denote a hyper complex or multicomplex system constructed from a group G as $HC(G)$ or $MC(G)$.

A class of extension of hyper complex systems can be formed by a Caylay-Dickson type construction.

Definition 2.3. Let HC_1 and HC_2 be two hyper complex systems with imaginary generators $i_k \in BHC_1$ and $j_l \in BHC_2$ respectively. A hyper complex system is called a Modified Cayley-Disckson product if its (vector) elements can be expressed in the form

$$w = \sum_{j_l \in BHC_2} a_l j_l = \sum_{j_l \in BHC_2} \left(\sum_{i_k \in BHC_1} a_{l,k} i_k \right) j_l,$$

where $a_l = \sum_{i_k \in BHC_1} a_{l,k} i_k$ are hyper complex numbers in HC_1.

We denote the Modified Cayley-Dickson (MCD) product by $HC_1 \otimes HC_2$. Here, we omit any condition on the multiplication of two symbols from the systems. It can be set commuting or anti-commuting as per needs.

Proposition 2.4. If H, K are proper subgroups of a group G with $G = HK$ and $H \cap K = \{1\}$, then $HC(G) = HC(H) \otimes HC(K).$

It is clear that $HC(n)\otimes HC(m)$ and $HC(m)\otimes HC(n)$ have rank mn, but may not be equal in general. The MCD product of two cyclic hyper complex systems need not be cyclic and the MCD product of more than two hyper complex systems is possible through recursion of Definition 2.3.

The MCD product is, in general, neither commutative nor associative. We shall, however focus our attention on commutative MCD products.

Definition 2.5. A multicomplex system is called basically cyclic if it can be expressed as a Modified Caylay-Dickson product of one or more cyclic multicomplex systems.

We state the following result concerning multicomplex systems of finite rank.

Proposition 2.6: A multicomplex system is of finite rank if and only if it is basically cyclic with finite ranked cyclic systems.

Proof. It is enough to prove the necessity. Let $G = BMC(n)\cup -BMC(n)$ be the group of symbols of a cyclic multicomplex system $MC(n)$. There is an imaginary symbol α such that $\alpha^{2n}=1$ or $\alpha^{n}=\pm 1$. Let p be the least integer such that $\alpha^{p}=\pm 1$. If $p=n$, the proposition is proved.

If $p<n$, consider the cyclic subgroup $H=\langle\alpha\rangle$ generated by α. By Lagrange's theorem, $n = pq$ for some integer q. Now, the set of cosets of H has q members. Select only one element from each different coset including the unit. The q elements form a subgroup K and we have $H\cap K=\{1\}$. Therefore, $G=HK$ and we have $MC(G)=MC(H)\otimes MC(K)$. If K is not cyclic, apply the same process on K repeatedly until it ends in finite steps.

The well known multicomplex systems obtained through MCD product are the bicomplex $C_2 = C\otimes C$ with rank 4 and in general C_n which is the MCD product of multiple copies of C and has rank 2^n.

Proposition 2.7. A multicomplex system with finite rank > 2 is not a division algebra.

Proof. Let $MC(n)$ be a multicomplex system with rank n > 2.

a. If $MC(n)$ is cyclic, with $\alpha^{n}=1$, then $(1-\alpha)(1+\alpha+\alpha^2+\alpha^3+\cdots+\alpha^{n-1})=0$ and hence the non zero multicomplex numbers $a=1-\alpha$ and $b=1+\alpha+\alpha^2+\alpha^3+\cdots+\alpha^{n-1}$ are the zero divisors.

b. If $\alpha^{n}=-1$, we have the following cases.

i. If n is odd, then we have $(1+\alpha)(1-\alpha+\alpha^2-\alpha^3+\cdots+\alpha^{n-1})=0$.

ii. If n is even, $n=2N$, with N odd, $(1+\alpha^2)(1-\alpha^2+\alpha^4+\cdots-\alpha^{2n-2})=0$.

iii. If N is even, repeat ii. until it ends in finite steps with an odd factor, or with $n=2^N$.

iv. If $n=2^N$, then it is isomorphic to C^N and hence is not a division algebra [6][7].

c. If $MC(n)$ is not cyclic, it is basically cyclic and has a sub algebra which is not a division algebra by the above arguments. Hence, $MC(n)$ is also not a division algebra by the zero divisors of the sub algebra extended to be in $MC(n)$ by appending zero coefficient terms.

For $n\le 2$, we have the division algebras real R and complex C, and therefore this case is excluded.■

Proposition 2.7 tells us that in order to have commutative division algebra, i.e. a field, with rank >2, one should look in to the classes of infinite dimensional spaces.

Now, we show by examples that the known existing multicomplex systems in the literature are particular cases of the above defined systems. Table 2.1 lists the examples with their definitions and their classes as defined above with references.

We see from Table 2.1 that the existing multicomplex systems in the literature are all basically cyclic.

Table 2.1: Known multicomplex systems

No	Symbol Definitions (existing)	Cyclic symbol	Name and Literature
1	$i^2=-1$	$i^2=-1$	Complex (Cyclic)
2	$i^2=1$	$i^2=1$	Split Complex (Cyclic)
3	$i_1^{\ 2}=-1, i_2^{\ 2}=-1, i_3^{\ 2}=1$	$i_1^{\ 2}=-1, i_2^{\ 2}=-1$	Bicomplex C-D product
4	$i_k^{\ 2}=-1, 1\le k<n$ and products	$i_k^{\ 2}=-1, 1\le k<n$	Multicomplex rank 2^n
5	$h^2=k, k^2=h, hk=1$	$h^3=1$	Circular[9] (Split Cyclic)
6	$1,\alpha,\alpha^2,\cdots,\alpha^{n-1}$	$\alpha^n=-1$	Multicomplex [6] (Cyclic)
7	$\alpha^2=-1, \beta^2=-1, \gamma^2=1,$ $\alpha\beta=-\gamma, \alpha\gamma=\beta, \beta\gamma=\alpha.$	$\alpha^2=-1, \beta^2=-1$	4D Circular complex [9] (Basically Cyclic)
8	$\alpha^2=1, \beta^2=1, \gamma^2=1,$ $\alpha\beta=\gamma, \alpha\gamma=\beta, \beta\gamma=\alpha.$	$\alpha^2=1, \beta^2=1$	4D Hyperbolic Complex [9] (Basically Cyclic)
9	$\alpha^2=\beta, \beta^2=-1, \gamma^2=-\beta,$ $\alpha\beta=\gamma, \alpha\gamma=-1, \beta\gamma=-\alpha.$	$\alpha^4=-1$	Planer Complex[9], Multicomplex[6] (Cyclic)
10	$\alpha^2=\beta, \beta^2=1, \gamma^2=\beta,$ $\alpha\beta=\gamma, \alpha\gamma=1, \beta\gamma=\alpha.$	$\alpha^4=1$	4D Polar Complex[9] (Cyclic)
11	$h_1^2=h_2, h_2^2=h_4, h_3^2=h_1, h_4^2=h_2,$ $h_1h_2=h_3, h_1h_3=h_4, h_1h_4=1,$ $h_2h_3=h_1, h_2h_4=h_1, h_3h_4=h_2.$	$h_1^5=1$	5D Polar Complex[9] (Cyclic)
12	$h_1^2=h_2, h_2^2=h_4, h_3^2=1, h_4^2=h_2,$ $h_ih_j=h_{i+j},\quad 1\le i<j\le 5$ with $h_{i+j}=h_{6-(i+j)}$ if $i+j>6.$	$h_1^6=1$	6D Polar Complex[9] (Cyclic)

A New Class Of Multicomplex Algebra: In this and the next sections, we define a new class of multicomplex system with some examples and their applications in partial differential equations.

If we relax the condition (1) in the C-D process, one may allow j^2 to couple with some coupling with other imaginary symbols.

Theorem 3.1. For a multicomplex system MC, with fundamental imaginary symbol t, set $Y=\{a+jb\equiv a(t)+jb(t)\,|\,a,b\in MC\}$ with addition and multiplication defined by $(a+jb)+(c+jd)=(a+c)+j(b+d)$ and $(a+jb)(c+jd)=(ac+ebd)+j(ad+bc)$, where $a,b,c,d\in MC$ is a multicomplex system if

$$j^2\in MC. \qquad (4)$$

Proof. Clear from algebraic manipulations.

Remark 3.2.

(a) The condition (4) on the theorem may look simple at first sight. But, notice that in the Cayley-Dickson process the condition is, in a sense, $j^2\in\{-1,1\}$. Here, we allow j^2 to couple with other symbols of MC while $j\notin MC$.

(b) We do not include the influence of conjugate a^* as in (2) in our extension to preserve the commutative property of the multiplication of the new system.

(c) One may also set, in the definition, $j^N\in MC$ while $j^k\notin MC, 1\le k<N$. However, we restrict our attention to the simplest case $j^2\in MC$ for clarity.

(d) The classical C-D process for multicomplex systems is now a sub process within our 'extended' C-D process.

If $MC(n)$ is cyclic with finite rank n then the new system Y has rank 2n. When j^2 is coupled with imaginary symbols from MC, we call the class a Coupled Multicomplex System and denote by $CMC(n,e)$, where $j^2=e(t)\in MC(n)$, n can be finite or infinite.

Note that in the new extended system, the basis symbols do not form a group anymore. In fact, the new symbol j does not form a group alone as it is coupled with symbols from $MC(n)$.

The question of division algebra may arise at this point. Under what condition is a coupled multicomplex system a division algebra? We have the following result.

Theorem 3.3. A coupled multicomplex system $MC(n,e)$ obtained from a cyclic multicomplex system over real is a division algebra if and only if

i. It is of infinite rank ($n=\infty$),

ii. The real part of e is negative. I.e., $e_0 < 0$.

Proof. From the discussion in Section 2, we see that the system must be of infinite rank. The two finite ranked division algebras, the real and complex, are not under coupled multicomplex systems as they do not have more than one imaginary symbol to couple.

Now, to prove sufficiency, a non-zero element $w = a(t) + jb(t) \in CMC(\infty, e)$ has an inverse $w^{-1} = \dfrac{a(t) - jb(t)}{a(t)^2 - eb(t)^2}$ if and only if the denominator is not zero. Hence, it is enough to prove the statement 'if $e_0 < 0$, then $a(t)^2 - eb(t)^2 = 0$ if and only if $a(t) = b(t) = 0$.'

For an element, $a(t) = \sum_{k=0}^{\infty} a_k t^k \in MC(\infty)$, define $A_n(t) = \sum_{k=n}^{\infty} a_k t^{k-n}$ so that we have $a(t) = A_0(t)$ and $A_n(t) = a_n + tA_{n+1}(t)$.

Let $e(t) = e_0 + tE_1(t)$ with $e_0 < 0$ and $a(t)^2 - eb(t)^2 = 0$. So,

$$(a_0 + tA_1(t))^2 - (e_0 + tE_1(t))(b_0 + tB_1(t))^2 = 0. \quad (5)$$

Equating the constant coefficients, we get $a_0^2 - e_0 b_0^2 = 0$ which means that $a_0 = b_0 = 0$.

Assume inductively that $a_k = b_k = 0$ for $k = 0, 1, \cdots, n-1$. Then equation (5) becomes,

$$(a_n + tA_{n+1}(t))^2 t^{2n} - (e_0 + tE_1(t))(b_n + tB_{n+1}(t))^2 t^{2n} = 0.$$

Again equating the coefficients of t^{2n}, we get $a_n^2 - e_0 b_n^2 = 0$ which gives $a_n = b_n = 0$. The converse is trivial.

Examples And Applications: Now, we give some simple examples of the new class of multicomplex systems.

Example 4.1. Let $MC(n,1)$ be a cyclic hyper complex system with imaginary generator t with $t^n = 1$. The system MCD product defined by setting $j^2 = -1 - t^2$. The multiplication is then defined by

$$(a + jb)(c + jd) = ac - (1 + t^2)bd + j(ad + bc),$$

where $a(t), b(t), c(t), d(t) \in MC(n,1)$ and are polynomials of degree < n.

This multicomplex system can be used to find general solutions of the system of partial differential equations of the form $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$ and $\dfrac{\partial^n u}{\partial x^n} = \dfrac{\partial^n u}{\partial y^n}$. In fact, for any n-times differentiable arbitrary function $u(.)$, the form $u \equiv u(x + yt + jz)$, we have $\dfrac{\partial^2 u}{\partial x^2} = u^{(2)}, \dfrac{\partial^2 u}{\partial y^2} = t^2 u^{(2)}, \dfrac{\partial^2 u}{\partial z^2} = j^2 u^{(2)}$, and $\dfrac{\partial^n u}{\partial x^n} = u^{(n)}, \dfrac{\partial^n u}{\partial y^n} = t^n u^{(n)}$. Then, clearly u satisfies the given two differential equations with $t^n = 1$ and $j^2 = -1 - t^2$. Expanding u with the aid of these relations, one sees that u is a $MC(n)$ hyper complex valued function of the hyper complex variable $w = x + ty + jz$.

If we write $u(w) = \sum_{k=0}^{n-1} t^k u_k(x,y,z) + j \sum_{k=0}^{n-1} t^k v_k(x,y,z)$, by the independency of the symbols, each of the real valued functions $u_k(x,y,z)$ and $v_k(x,y,z)$ will satisfy the given system of partial differential equations.

Example 4.2 (A hypercomplex field). If we remove the condition $t^n = 1$ in Example 4.1, we have a multicomplex system of infinite rank $CMC(\infty, -1 - t^2)$. One can easily check that this hyper complex system is a division algebra, forming a field.

The function $u \equiv u(x + yt + jz)$, will satisfy the Laplace equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$ with $1 + t^2 + j^2 = 0$. Therefore, expanding u as $u = \sum_{k=0}^{\infty} u_k(x,y,z) t^k + j \sum_{k=0}^{\infty} v_k(x,y,z) t^k$, each function $u_k(x,y,z)$ and $v_k(x,y,z)$ will satisfy the

Laplace equation. In particular, if one chooses $u \equiv (x + yt + jz)^l$, $l \in N$, expanding this we see $2l+1$ functions satisfying the Laplace equation for each 'mode' l. This set of basis solutions have been obtained earlier[10][11]. But, to our knowledge here is where a complex system is used to obtain this basis solutions. Further analysis on this space of solutions can be seen in [12][13] .

Example 4.3 (A coupled system for Wave equations). Let $MC(\infty) \otimes MC(\infty)$ be an infinite rank multicomplex system with imaginary generators α and β. A coupled multicomplex system $CMC(\infty, e)$ is formed with $j^2 = e(\alpha, \beta) = 1 - \alpha^2 - \beta^2$. Although it is not a division algebra by means of Theorem 3.3, it is still useful for solutions of PDEs.

The function $u \equiv u(ct + \alpha x + \beta y + jz)$, will satisfy the 3D wave equation $\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$ with j, α, β satisfying $j^2 = 1 - \alpha^2 - \beta^2$. Expanding u, as $u = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} u_{k,l}\alpha^k \beta^l + j\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} v_{k,l}\alpha^k \beta^l$, the functions $u_{k,l}(x, y, z)$ and $v_{k,l}(x, y, z)$ will satisfy the 3D wave equation.

Conclusion : We proposed an extension of the Ceyley-Dickson process to include new classes for multicomplex systems. The new classes have applications in constructing general solutions and basis of solutions to partial differential equations. These are demonstrated through examples.

To the knowledge of the author, these new classes of multicomplex systems have not appeared before in the literature.

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