A BRIEF SURVEY OF APPROXIMATION PROPERTIES

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Abstract: Analytic properties of invariant approximation property, studies analytic techniques from operator theory that encapsulate geometric properties of a group. we investigate its links to the completely bounded approximation property (CBAP), the Invariant approximation property(IAP), the approximation property (AP), the operator space approximation property (OAP), Strong Invariant approximation property (SIAP) and exactness. We then use this to show the following groups have invariant approximation property: Amenable groups, Hyperbolic groups, CAT(0)-cubical groups, $SL2(Qp)$ and Sp(1, n).

Keywords: Weakly amenable, Approximation property, Invariant approximation property.

Introduction: Uniform Roe C^*- algebras provide, among other things, a link between coarsegeometry and C^*- algebra theory. Let G be a discrete group. The reader is referred to Roe [12], Kannan [9], Jolissaint [8], Brodzki [4] and Anantharaman-Delaroche [1] for the details on the invariant approximation property and the coarse geometry.

The purpose of this note is to provide an illustration of an interesting and nontrivial interaction between analytic and geometric properties of a group. We provide a short survey of approximation properties of operator algebras associated with discrete groups. We give a general exposition of approximation properties which were initiated by Grothendieck [3]. His fundamental ideas have been applied to the study of groups and these noncommutative approximation properties have played a crucial role in the study of von Neumann algebras and C^*-algebra. Some weaker conditions (i.e., weak amenability and the approximation property) for locally compact groups have been studied by Haagerup and Kraus [7].We recall basic definitions of approximation properties. Let C^*-algebra A said to have the *completely bounded approximation property* (CBAP) if there is a positive number C such that the identity map on A can be approximated in the point norm topology by a net $\{\varphi_\alpha\}$ of finite rank completely bounded maps whose completely bounded norm are bounded by$C$, that is if there exists a net of finite-rank maps $\{\varphi_\alpha\}: A \longrightarrow A$ such that $\|\varphi_\alpha\|_{cb} \leq C$ for some constant C and $\varphi_\alpha \longrightarrow id_A$ in the point-norm topology on A. The infimum of all values of C for which such constants exist is denoted by $\lambda_{cb}(A)$ [7]. We say that G is *weakly amenable* if there is a net $\{\varphi_\alpha\}$ in $A(G)$ and a constant C such that

$\|\varphi_\alpha v - v\| \longrightarrow 0 \quad \forall v \in A(G)$ and such that $\|\varphi_\alpha\|_{M_{\varphi_\alpha}} \leq C \quad \forall\alpha$. There are many other interesting approximation properties for C^*-algebra. It is shown in [7] that a C^*-algebra A has the *operator approximation property* (OAP) if there exists a net of finite-rank maps $T_\alpha: A \longrightarrow A$ such that $T_\alpha \to id_A$ in the stable point-norm topology. The discrete group G has the approximation property (AP) if there is a net $\{\varphi_\alpha\}$ in $A(G)$ such that

$M_{\varphi_\alpha} \longrightarrow id_{A(G)}$ in the stable point-norm topology on $A(G)$ [7].

Haagerup and Kraus [7] show that a discrete group G has the *approximation property* (AP) if and only if $C_r^*(G)$ has the OAP.

We have the following important result from Haagerup and Kraus [7].

Theorem 1.1: Let G be a discrete group. Then the following are equivalent:

- G has the AP,
- $C_r^*(G)$ has the OAP,
- $C_r^*(G)$ has the SOAP.

Example 1.2: The following groups have AP. This implies that these groups have the OAP, and thus also SOAP:

- $SL(2,\mathbb{Z})$
- $\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})$

Definition 1.3:

A C^*-algebra A is *exact* if, given any exact sequence

$$0 \longrightarrow J \longrightarrow B \longrightarrow C \longrightarrow 0$$

of C^*-algebras, the sequence

$$0 \longrightarrow A \otimes_{min} J \longrightarrow A \otimes_{min} B \longrightarrow A \otimes_{min} C \longrightarrow 0$$

is again exact.

Definition 1.4: We say that a discrete group G is *exact* if and only if $C_r^*(G)$is an exact C^*algebra.

Example 1.5: Kirchberg and Wassermann [10] show that if a $C_r^*(G)$ has the CBAP then G is exact. On the other hand a group G is weakly amenable if and only if it has the CBAP [7] and so all weakly amenable groups are exact.

Example 1.6: The following are examples of exact group:

- Linear groups [6]
- Hyperbolic groups [11]
- Coxeter groups

- Countable subgroups of almost connected Lie groups [6]

Coarse geometry is the study of the large scale properties of spaces. The notion of large scale is quantified by means of a coarse structure.

Example 1.10: Let G be a finitely generated group. Then the bounded coarse structure associated to any word metric on G is generated by the diagonals $\Delta_g = \{(h, hg): h \in G\}$,as g runs over G. We shall denote the finite propagation kernels on X by $A^{\propto}(X)$.

Definition 1.11: The *uniform Roe algebra* of a metric space X is the closure of $A^{\propto}(X)$ in the algebra $B(\ell^2(X))$ of bounded operators on X.

If a discrete group G is equipped with its bounded coarse structure introduced in Example 1.10 then one can associated with it to uniform Roe algebra $C_U^*(G)$ by repeating the above. We next recall some basic facts about uniform Roe algebra and metric property of a discrete group.

Next we recall the following definitions. Let X be a discrete metric space.

Definition 1.12: We say that discrete metric space X has *bounded geometry* if for all R there exists N in $\mathbb{N}$ such that for all $x \in X$ $|B_R(x)| < N$, where $B_R(x) = \{x \in X: d(y, x) \leq r\}$.

Definition 1.13: A kernel $\varphi: X \times X \longrightarrow \mathbb{C}$

1. is *bounded* if there, exists $M > 0$ such that $|\varphi(s,t)| < M$ for all $s, t \in X$
2. has *finite propagation* if, there exists $R > 0$ such that $\varphi(s,t) = 0$, $d(s,t) > R$

Let $B(X)$ be a set of bounded finite propagation kernels on $X \times X$. Each such φ defines a bounded operator on $\ell^2(X)$ via the usual formula for matrix multiplication

$\varphi * \psi = \sum_{r \in G} \varphi(s,r)\psi(r)$ for $\psi \in \ell^2(X)$.

Next, we had [10] that the operator associatedwith a bounded kernel is bounded.

Lemma 1.14: Let X be bounded geometry metric space. An operator associated with a bounded finite propagation kernel is bounded.

2. Invariant Approximation property: In this section we will give the definition of invariant approximation property. A discrete group G has a natural coarse structure which allows us to define the uniform Roe algebra $C_U^*(G)$. A group G can be equipped with either the left or right-invariant of the metric. A choice of one of the determines whether $C_\lambda^*(G)$ or $C_\rho^*(G)$ is a subalgebra of the uniform Roe algebra of G as we now explain. If the metric of G is right - invariant then

$$C_\lambda^*(G) \subseteq C_U^*(G).$$

Let d_1 be the right-invariant metric on G.

$$d_1(x,y) = d_1(xg, yg) \forall g \in G.$$

The operator $\lambda(g)$ is given by the matrix.

$$A_g^\lambda(x,y) = \begin{matrix} 1 & , if\ x = yg. \\ 0 & , otherwise. \end{matrix}$$

Note that $A_g^\lambda(x,y)$ is right-invariant:

$$A_g^\lambda(xt, yt) = \begin{Bmatrix} 1 & xt = ygt \\ 0 & otherwise. \end{Bmatrix}$$

Therefore: $A_g^\lambda(xt, yt) = A_g^\lambda(x,y)$. If the metric on G is right-invariant, $A_g^\lambda(x,y)$ is of finite propagation and $A_g^\lambda(x,y) \in C_U^*(G)$ since $A_g^\lambda(x,y)$ is non-zero when $y^{-1}x = g$ and so $d_1(x,y) = d_1(xg, yg)$. Hence any element of $C[G]$ will finite propagation and this assignment extends to an inclusion $C_\lambda^*(G) \hookrightarrow C_U^*(G.)$. Similarly we can show that if the metric on G is left-invariant then

$$C_\rho^*(G) \subseteq C_U^*(G).$$

Let us now choose a right invariant metric for G so that $C_\lambda^*(G) \hookrightarrow C_U^*(G)$. The right regular representation ρ gives the adjoin action on $C_U^*(G)$ defined by $Ad\rho(g)T = \rho(g)T\ \rho(g)^{-1}$ for all $t \in G$, $T \in C_U^*(G)$. The above remarks show that elements of $C_\lambda^*(G)$ are invariant with respect to this action and so $C_\lambda^*(G)$ is contained in invariant subalgebra $C_U^*(G)^G$.

The following important result as given in [9].

Lemma 2.1: If $T \in C_U^*(G)$ has kernel $A(x,y)$, then $A\rho(t)T$ has kernel $A(xt, yt)$.

In general, if $T \in C_U^*(G)$ then $\forall x, y \in G$,

$\langle Ad\rho(t)T\delta_x, \delta_y \rangle = \langle T\delta_{xt}, \delta_{yt} \rangle$.

So the operator T is $Ad\rho -$ invariant if and only if

$$\langle Ad\rho(t)T\delta_x, \delta_y \rangle = \langle T\delta_{xt}, \delta_{yt} \rangle ,$$

$\forall x, y \in X, \forall t \in G$.

We now define the invariant approximation property: (IAP)

Definition 2.2: We say that G has the *invariant approximation property* (IAP) if

$$C_U^*(G)^G = C_\lambda^*(G).$$

3. CBAP, AP and IAP : We note also the following results.

Lemma 3.1: If G is weakly amenable, then G has the AP.

Lemma 3.2: If G has the AP, then G has the IAP.

Proof: By Zacharias Theorem [8], the invariant translation approximation property with coefficients implies the one without coefficients. This means:

$$C_U^*(G, \mathbb{C})^G = C_U^*(G)^G \otimes \mathbb{C} = C_\lambda^*(G) \otimes \mathbb{C}$$

and therefore

$$C_U^*(G)^G = C_\lambda^*(G).$$

Thus G has IAP.

We are now ready to prove the following proposition.

Proposition2.5: The following implications hold for a discrete group:

$$CBAP \Rightarrow AP \Rightarrow IAP$$

Proof: By Theorem [7] if G is discrete group, then G is weakly amenable if and only if $C_r^*(G)$ has the CBAP. But G is weakly amenable implies that G has

AP. As now we use Lemma 3.1 and Lemma 3.2 to conclude that

$CBAP \Rightarrow AP \Rightarrow IAP$.

Remark 3.4: The converse of the first implication does not holds: a counter example is given by $\mathbb{Z}^2 \rtimes SL(2, \mathbb{Z})$. Since AP is preserved by semi-direct products [7], this group has the AP. But Haagerup [7] proved that it does not have the CBAP.

The following groups are all weakly amenable. This implies that these groups have the AP, and thus also IAP: The following groups are all weakly amenable. This implies that these groups have the AP, and thus also IAP:

- Amenable groups
- Hyperbolic groups [11]
- CAT(0)-cubical groups [13]
- $SL_2(\mathbb{Q}_p)$ [2]

We also show that every amalgamated product of a countable collection of discrete amenable groups over a compact open subgroup has the invariant approximation property.

Theorem 3.6 [19]: Let $(G_i)_{i\in I}$ be weakly amenale discrete groupswith constant 1. Then $G = *_{i\in I} G_i$ is also weakly amenablewith constant 1.

Theorem 3.7: Amalgamated products of amenable groups has IAP

4 .Joachim Zacharias's IAP with coefficients : In this section we will give definition of the strong invariant approximation property. Let $S \subseteq B(\mathbb{H})$ be a closed subspace. We define the operator space $C_u^*(G, S)$ as the closure of finite width matrices $[a_{s,t}]_{s,t\in G}$,where $a_{s,t\in S}$ and $[a_{s,t}]_{s,t\in G}$ is uniformly bounded for all $s, t \in S$ acting on $\ell^2(G)\otimes\mathbb{H}$

Next, we define the set of fixed points of $C_u^*(G, S)^G$.

We define

$C_u^*(G, S)^G = \{T \in C_u^*(G, B(\mathbb{H})) \colon \mathrm{ad}(\rho_t\otimes \mathrm{id})T = t \text{ for all } t \in G\}$

We now define Joachim Zacharias's IAP with coefficients (SIAP):

Definition 4.1: We say that a discrete group G has the strong invariant translation approximation property (SIAP) if for any closed subspace S of the compact operators $\mathbb{K}$ (on $\ell^2(\mathbb{N})$. We have an isomorphism

$$C_u^*(G, S)^G = C_\lambda^*(G)\otimes S.$$

Next, we show the following:

Proposition 4.2: SIAP implies IAP for discrete exact groups.

Proof : Let G be a group with SIAP and $S = \mathbb{C}$, we have that

$$C_u^*(G, \mathbb{C})^G = C_\lambda^*(G)\otimes\mathbb{C}$$

But

$$C_u^*(G\otimes\mathbb{C})^G = C_u^*(G)^G\otimes\mathbb{C}$$

so that $C_u^*(G)^G\otimes \mathbb{C} = C_\lambda^*(G)\otimes\mathbb{C}$

This implies $C_u^*(G)^G = C_\lambda^*(G)$.

Conclusion: The *uniform Roe algebra* $C_U^*(G)$ is the C^*- algebra completion of the algebra of bounded operators on $\ell^2(X)$ which have finite propagation. In other words: The *uniform Roe algebra* $C_U^*(G)$ is the norm closure in $B(\ell^2(G))$ of the $*$-subalgebra formed by the operator $Op(k)$, where k ranges over the bounded kernel with finite propagation. This means that:

$$C_U^*(G) = \overline{Op(k)}$$

where k is finite propagation kernel on G. The reduced $C^* -$ algebra $C_r^*(G)$ is naturally contained in $C_U^*(G)$. We also consider some of the elementary concepts associated with coarse spaces. Let G be a discrete group. We say that the uniform Roe algebra $C_U^*(G)$ is the C^*-algebra completion of the algebra of bounded operators on $\ell^2(X)$ which has finite propagation. According to Roe [12] G has the invariant approximation property (IAP) if

$$C_\lambda^*(G) = C_U^*(G)^G$$

References :

1. C.Anantharaman-Delaroche, Amenable correspondences and approximation properties for von Neumann algebras, Pacific J. Math. 171 (1995), no. 2, 309–341.
2. M. Bo˙zejko and M. A. Picardello, Weakly amenable groups and amalgamated products, Proc. Amer. Math. Soc. 117 (1993), no. 4, 1039–1046.
3. J. Brodzki and G. A. Niblo, Approximation properties for discrete groups, C^*- algebras and elliptic theory, Trends Math., Birkh¨auser, Basel, 2006, pp. 23–35.
4. Erik Guentner, Nigel Higson, and Shmuel Weinberger, The Novikov conjecture for linear groups, Publ. Math. Inst. Hautes ´Etudes Sci. (2005), no. 101, 243–268.
5. U.Haagerup and J.Kraus, Approximation properties for group C^*-algebras and group von Neumann algebras, Trans. Amer. Math. Soc. 344 (1994), no. 2, 667–699.
6. P. Jolissaint, Rapidly decreasing functions in reduced C^*-algebras of groups, Trans. Amer. Math. Soc. 317 (1990), no. 1, 167–196.
7. K. Kannan, On approximation properties of group C^* - algebras, University of Southampton, School of Mathematics,Doctoral Thesis,116pp., 2011.
8. K.Kannan, Uniform Roe Algebras as Crossed Product, JMSO-GSTF, 2013 no 2,
9. K.Kannan,Discrte group with Rapid Decay property JMSO-GSTF, 2013 no 2,
10. K. Kannan , Weakly amenable and invariant

approximation property, Proc. of the International Conference on Mathematics (ICMIT),India,.

11. E.Kirchberg and S wassermann, Exact groups and continuous bundles of C^*-algebras, Math. Ann. 315 (1999), no. 2, 169–203.
12. N. Ozawa, Weak amenability of hyperbolic groups, Groups Geom. Dyn. 2 (2008), no. 2, 271–280.
13. J. Roe, Lectures on coarse geometry, University Lecture Series, vol. 31, American Mathemat ical Society, Providence, RI, 2003.
14. A. Valette, Weak amenability of right-angled Coxeter groups, Proc. Amer. Math. Soc. 119 (1993), no. 4, 1331–1334.

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