ROLLE'S AND MEAN VALUE THEOREMS: MEETING POINTS AND CONTRASTS

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Abstract : This paper examines the Rolle's theorem (RT) and mean value theorem (MVT), draws connection and contrast between both theorems for functions. The mean value theorem also known as the average slope theorem is a special case of the RT. It is well established fact that differentiability and continuity properties are the essential features of all functions that are valid for both theorems. From our discovery, as contrast, both theorems do not always hold for the same function in same given intervals as readily and explicitly enumerated in some worked functions exemplified in this paper. But notably in some cases, if the interval of continuity and differentiability is changed for an already considered function that only MVT holds for, then RT will hold for such function.

Keywords : Continuous, Function, Mean Value Theorem (MVT), and Rolle's Theorem (RT).

Introduction : The Rolle's theorem (RT) and the Mean value theorem (MVT) are very useful theorems in mathematical analysis. The former was postulated by Michael Rolle (1652-1719) who was a great 17^{th} century mathematician. His work on function is very critical in mathematical analysis. The MVT was formulated by Joseph Louis Lagrange (1736-1813), an Italian Professor of mathematics. It is a special case of the RT. The proof of the theorems are not shown here (see [2], [3] and [9]) but categorically stated here. Theorem on Local Extrema is an essential tool for the derivation of the RT. If $f(c)$ is a local extremum, then either f is not differentiable at c or $f'(c)=0$. That is, at a local maximum or minimum, f either has no tangent, or f has a horizontal tangent there. In this paper, we shall examine both RT and MVT with application to some functions to draw out their meeting point and differences.

Continuity And Differentiability Of Functions : A function f: D $\subseteq \Re \rightarrow \Re$ is said to be continuous at the point x $\in$D if $\lim_{x\to x_o} f(x)=f(x_0)$. This definition can be simply modified as f will be continuous at a point x_0 provided $f(x_0)$ is defined, $\lim_{x\to x_o} f(x)$ exists and $\lim_{x\to x_o} f(x)=f(x_0)$. These conditions are just polishing the basic definition of continuity. The simpler meaning of continuity is relating to a line or curve along which the difference between function values at any two points within a given interval will approach zero if the interval is decreased sufficiently. A function f is continuous at x_0 if and only if for a given ε > 0 there exists δ > 0 such that $|x-x_0|<\delta$ implies that $|f(x)-f(x_0)|<\varepsilon$. This is basically the Cauchy's analytical definition and contribution to continuity of a function.

Let $f:[a,b]\rightarrow \Re$ and let $x\in[a,b]$, then f is said to have a derivative (or to be differentiable) at x_0 if $\lim_{x\to x_o}\dfrac{f(x)-f(x_o)}{x-x_o}$ exists.

Remark 1: These two definitions above are the very important ingredients in the RT and MVT.

Rolle's Theorem (Rt) : Let the function $f(x)$ be continuous in $[a,b]$ and differentiable in (a,b). Suppose that $f(a)=f(b)$ = 0; then there exists a point $x\in(a,b)$ such that $f'(x)=0$. Put in another way, RT says that if f is a differentiable function, then between any two solutions of the equation $f(x)=0$, there is a point c where $f'(c)=0$. This basic result helps us to identify intervals where an equation has a unique solution. We shall illustrate the theorem using the function $f(x)$ = $2x^3 + x^2 - 4x - 2$ to establish the value c∈(-√2, √2) for which $f'(c)=0$. Now, by comparing the problem with the RT we have a = -√2 and b = √2 and put x = c. This is a polynomial of degree 3, it is differentiable and continuous everywhere. Let c∈[-2,2] such that -2 < c < 2. Then, $f(c)$ = $2c^3 + c^2 - 4c - 2$ with $f'(c)$ = $6c^2+2c-4$ = 0. One readily notes here that $f(a)=f(b)$ and solving yields $c=-1$ or $c=\dfrac{2}{3}$ which obviously established the existence of RT for the considered function. Now, suppose we have the interval above as $(-2,2)$ for the same function, then we readily observe that RT's hypotheses will not all be satisfied as $f(-2)\neq f(2)$ and yet there is the existence of a $c\in(-2,2)$. This then leads us to the MVT.

The Mean-Value Theorem (MVT) : The theorem states that if $f(x)$ is continuous in $[a,b]$ and differentiable in (a,b), then ∃ (there exists) c∈ (a,b) ∍ (i.e. such that) $f'(c)=\dfrac{f(b)-f(a)}{b-a}$. Now, if f

satisfies the hypotheses of the RT, then the MVT also applies. The proof of this is trivial and easily derived from that of the RT (See [4]). The MVT is an existence theorem whose interest is typically on the existence of a $c\in(a,b)$. This theorem will be explained using the same function $f(x)$ = $2x^3 + x^2 - 4x - 2$ as above with the interval $(-2,2)$. Here, we obtain $f(-2) = -6 \neq f(2) = 10$. Hence, by the MVT, $f'(c) = \dfrac{f(b) - f(a)}{b - a} \Rightarrow 6c^2 + 2c - 4 = 4$ and solving yields $c = 1$ or $c = -\dfrac{4}{3}$ which readily establishes the validity of the theorem for the function f. Similarly, we consider a polynomial function $f(x) = x^3$ on the interval $[1,3]$. f is a polynomial and so continuous everywhere. For any x, $f'(x) = 3x^2$. So, f is continuous on [1, 3] and differentiable on (1, 3). We readily obtain $c = \sqrt{\dfrac{13}{3}}$ since $\dfrac{f(b) - f(a)}{b - a} = 13$ and $f'(c) = 3c^2$ where the obtained c is in $[1,3]$. So the Mean Value theorem applies to f.

Meeting Point And Contrasts : Both theorems, RT and MVT have a meeting point with common features such as the function is differentiable in same interval (a, b) and continuous in [a, b]. In both theorems, there is the existence of $c\in(a,b)$ and $f'(c)$ is the same for every c. The major difference is the fact that for RT, $f(a) = f(b)$ and for MVT, $f(a) \neq f(b)$. Thus, it should be noted that the MVT is a special case of the RT where $f^1(c) = \dfrac{f(b) - f(a)}{b - a} = 0$ since $f(a) = f(b)$.

Thus, by illustration, it is possible that RT may hold for a function under a certain interval and if such interval is changed, it may be only MVT that holds for such function as illustrated by the function $f(x) = 2x^3 + x^2 - 4x - 2$ used above with the interval $(-\sqrt{2}, \sqrt{2})$ and later to $(-2,2)$. This eventually makes us know that the MVT is just a special case of the RT but where $f(a) = f(b)$, the existence of $c\in(a,b)$ was just established.

For further illustration, the function $f(x) = x^3$ in the interval [1, 3] considered above established that MVT holds for f. In case, the interval is changed for example, to [-2, 2], we readily see that $f(-2) = -8 \neq 8 = f(2)$ which defiles the RT hypothesis even though the function is differentiable and continuous in the considered interval [-2, 2]. Now, for the MVT, we readily obtain

$$f'(c) = 3c^2 = 4 = \frac{f(b) - f(a)}{b - a}$$

from which we get $c = \pm\dfrac{2}{\sqrt{3}}$ and both $c = \dfrac{2}{\sqrt{3}}$ and $c = -\dfrac{2}{\sqrt{3}}$ are in the interval [-2, 2]. Thus, MVT readily applies to the function at both intervals except RT.

Remark 2: The application of RT and MVT is not only limited to algebraic functions such as the polynomials considered in our worked examples but they are also applicable to trigonometric functions alike. For illustration purpose, if $f(x) = \sin x$ and considered in the interval $[0,\pi]$. Obviously, $f(0) = f(\pi)$ and $f'(c) = \cos c$ from which $c = \dfrac{\pi}{2} = 90^o \in [0,\pi]$. f is differentiable and continuous in the interval $[0,\pi]$. But now, if we say $f(x) = \tan x$, certainly there is no $c \in [0,\pi]$ in which $f'(c) = 0$ even though $f(0) = f(\pi)$. Then, shall we say this is a counter-example to RT? Obviously no! We readily note that $f(x) = \tan x$ is not continuous over $[0,\pi]$ if in particular we consider $\dfrac{\pi}{2}$.

Conclusion : From the foregoing, we see that both theorems, RT and MVT have a lot of relevance in mathematical analysis. Differentiability and continuity properties are the essential features of all functions that both theorems apply to. It is possible that RT may be applicable to a function under a certain interval and if such interval is changed, it may be only MVT that applies to such function.

References :

1. Anton, H. (2005). Elementary Linear Algebra. Application Version, 9th Ed. Willey International, London.
2. Balogun, F. O.(2006). Mathematical Methods. Kokoka Brothers Ltd. Nigeria.ISBN 978-37948-9-2.
3. http://de2de.Synechism.org. Rolle's Theorem and Mean Value Theorem.
4. http://math.berkeley.edu/~buehler/A.../workshee t21-solution.pdf.
5. http://math.slu.edu/~clair/mcmc/w-2010-IVT-

MVT-solutions.pdf.
6. Mohamed, A. K. and Helmut, K. (2010). The Mean-Value Theorem. Retrieved from
7. http:/www.sosmath.com.
8. Stroud, K. A. (2001). Engineering Mathematics, 8th Ed. Anthony Rowe Ltd. Wiltshire.
9. Webber, J. B. (1982). Mathematical analysis, 4th Ed. Harpir Institute, London.
10. William, A. L. (2012). Rolle's Theorem and the Mean Value Theorem. Retrieved from
11. http://www.math.hawaii.edu/~bill/MEANnAPPS/12Rollemean.pdf.

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