A NEW FUZZY SUBGROUP ON 3-DEGREE SYMMETRIC GROUP

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Abstract : In this study, a new fuzzy subset is defined on the 3-degree symmetric group and investigated some of its properties. A fuzzy set $\mu : S_3 \to [0,1]$ is defined in a new manner which forms a fuzzy subgroup on S_3. This fuzzy subgroup μ on 3-degree symmetric group forms a fuzzy coset and pseudo fuzzy coset. But this fuzzy subgroup μ is not a fuzzy normal subgroup. Finally, conjugate fuzzy subgroups of μ in S_3 are obtained.

Keywords : Conjugate fuzzy subgroups, 3-degree symmetric group, fuzzy coset, fuzzy normal subgroups.

Introduction : In 1965, Lofi A. Zadeh introduced the concept of fuzzy set. In the last five decades, the fuzzy concept is developed in the field of fuzzy algebra. Fuzzy algebra is used as a tool to study and understand the structure of the algebra explicitly. In [1] Dae Sing Kim said, "Rosenfeld introduced a fuzzy subgroup in [6]. Das characterized fuzzy subgroups by their level subgroups and Liu defined the fuzzy normality of a fuzzy subgroup as in [2]". Reference [5] included 570 research papers in the bibliography, which are closely related to fuzzy set theory and its applications listed by Kandel. Vasandha Kandasamy's Smarandache fuzzy algebra included the basic concepts in fuzzy algebra with suitable examples in [7]. Jin Bai Kim, studied fuzzy symmetric subgroups and obtained some of its properties as in [4]. The concept of fuzzy algebra is used in various fields of Mathematics, Engineering, Electrical Networks and Real World Problems etc. In this paper, a new fuzzy subgroup is obtained on the 3-degree symmetric group and properties of these fuzzy subgroups are discussed through the lemmas. Finally this paper concludes with a new fuzzy subgroup which is conjugate to itself and the other conjugate subgroups of μ in G are also obtained.

Preliminaries: Let us recall some basic definitions which are related to this paper.

Definition 1 [3]: If $\sigma : S \to T$ and $\tau : T \to U$ then the composition of σ and τ (also called their product) is the mapping $\sigma \circ \gamma : S \to U$ defined by means of $s(\sigma \circ \gamma) = (s\sigma)\tau$ for every $s \in S$.

Definition 2 [3]: Let S is a finite set having n elements {1,2,...,n}. If S_n be the group of all one-one mappings of the set S onto itself, under the product which we defined in definition 1 then S_n is called "symmetric group" of degree n and $o(S_n) = n!$

Definition 3 [7]: Let X be a non-empty set. A fuzzy set (subset) μ of the set X is a function $\mu : X \to [0,1]$.

Definition 4 [7]: Let G be a group. A fuzzy subset μ of a group G is called a fuzzy subgroup of the group G if (i) $\mu(xy) \ge \min\{\mu(x), \mu(y)\}$ $\forall$ x, y $\in$ G,

(ii) $\mu(x^{-1}) = \mu(x)$ for every x $\in$ G.

Definition 5 [7]: Let μ be a fuzzy subgroup of a group G. For any a $\in$ G, aμ defined by $(a\mu)x = \mu(a^{-1}x)$ for every x $\in$ G is called the fuzzy coset of the group G determined by a and μ.

Definition 6 [7]: Let μ be a fuzzy subgroup of a group G and a $\in$ G. Then the pseudo fuzzy coset $(a\mu)^P$ is defined by $(a\mu)^P(x) = p(a)\mu(x)$ for every x $\in$ G and for some $p \in P$.

Definition 7 [7]: A fuzzy subgroup A of G is called normal if $A(x) = A(y^{-1}xy)$ for all x,y $\in$ G.

Definition 8 [7]: Let λ and μ be two fuzzy subgroups of a group G. Then λ and μ are said to be conjugate fuzzy subgroups of G if for some g $\in$ G, $\lambda(x) = \mu(g^{-1}xg)$ for every x $\in$ G.

Theorem 9 [7]: A fuzzy subset μ of a group G is a fuzzy subgroup of the group G if and only if $\mu(xy^{-1}) \ge \min\{\mu(x), \mu(y)\}$ for every x, y $\in$ G.

Fuzzy Subset On 3-Degree Symmetric Group: Let us consider a non empty set S = {1, 2, 3}, we define S_3 to be the set of all one-one mappings of the set S on to itself.

Let the elements of the group

$S_3 = \{p_1, p_2, p_3, p_4, p_5, p_6\}$, where

$p_1 = (1),\ p_2 = (1,\ 2),\ p_3 = (1,\ 3),$

$p_4 = (2, 3),\ p_5 = (1, 2, 3),\ p_6 = (1, 3, 2).$

Clearly o(S) = 3 and $o(S_3) = 3!$

Obviously, S_3 is a group under the product defined in definition 1.

Let us take G = S_3.

Now define a new fuzzy set $\mu : G \rightarrow [0,1]$ as follows, $\forall\ x \in G,$

$$\mu(x) = \begin{cases} (o(S)-1)/o(S_n) & if \quad x = p_1 \\ (o(S)-2)/o(S_n) & if \quad x = p_2 \\ (o(S)-3)/o(S_n) & if \quad x = p_3 \quad (1) \\ (o(S)+1)/o(S_n) & if \quad x = p_4 \\ (o(S)+2)/o(S_n) & if \quad x = p_5 \\ (o(S)+3)/o(S_n) & if \quad x = p_6 \end{cases}$$

By this definition the membership function $\mu(x)$ is in the interval [0,1]. If $x = p_1, p_2, p_3, p_4, p_5$ & p_6 respectively then the actual value of the membership function defined by (1) is given below, $\mu(x) = \{\ 0.33,\ 0.17,\ 0,\ 0.67,\ 0.83,\ 1\ \}.$

Fuzzy Subgroup On 3-Degree Symmetric Group:

Theorem:1 A fuzzy subset μ on G defined by the equation (1) is a fuzzy subgroup of the 3-degree symmetric group.

Proof: Let μ be a fuzzy subset of a 3-degree symmetric group. By the theorem explained in [7], "A fuzzy subset μ of a group G is a fuzzy subgroup of the group G if and only if $\mu(xy^{-1}) \geq \min\{\mu(x), \mu(y)\}$ for every x,y$\in$ G."

Using this statement, to prove the theorem it is enough to prove that the fuzzy set satisfies the following inequality $\mu(xy^{-1}) \geq \min\{\mu(x), \mu(y)\}$ for every x,y$\in$ G.

Let us choose two arbitrary elements x = p_2 = (1, 2), y = p_5 = (1, 2, 3) in G. y^{-1} = (1, 3, 2) = p_6.

xy^{-1} = (2, 3) = p_4, $\mu(p_4) = 0.67$,

$\mu(x) = \mu(p_2) = 0.17$, $\mu(y) = \mu(p_5) = 0.83.$

$\min\{\ \mu(x), \mu(y)\} = 0.17$.

From the above calculations, clearly the inequality $\mu(xy^{-1}) \geq \min\{\mu(x), \mu(y)\}$ is satisfied for x = p_2, y = p_5. Likewise the inequality is satisfied for every x, y $\in$ G. Hence μ is a fuzzy subgroup of the 3-degree symmetric group.

Now the properties of the fuzzy subgroup μ on G defined by equation (1) are explained through the following three lemmas.

Lemma:2 If μ be a fuzzy subgroup on G defined by the equation (1) then for any a$\in$ G, $a\mu$ defined by $(a\mu)x = \mu(a^{-1}x)\ \ \forall\ x \in G$ is the fuzzy coset of a group G.

Proof: Let μ be a fuzzy subgroup of a group G defined by the equation (1). By the definition of fuzzy coset of a group G, for any a$\in$ G, $a\mu$ is defined by $(a\mu)x = \mu(a^{-1}x)\ \forall\ x \in G.$

Let us choose the value of 'a' as $p_4 = (2,\ 3)$. On computation the value of $a^{-1} = p_4^{-1} = (2,\ 3)$. For every $x \in G$ the value of $(a\mu)x$ and $\mu(a^{-1}x)$ are calculated and listed in the Table I.

From the below Table I, the value of $(a\mu)x$ and $\mu(a^{-1}x)$ are coincide for every $x \in G,$ This shows that the fuzzy coset $a\mu$ is in the 3-degree symmetric group G. This concludes the proof of the lemma.

Table I. Fuzzy coset property on G

Value of 'a' = a^{-1} = p_4			
Elements in G	$(a\mu)x$	$a^{-1}x$	$\mu(a^{-1}x)$
p_1	0.67	p_4	0.67
p_2	0.83	p_5	0.83
p_3	1	p_6	1
p_4	0.33	p_1	0.33
p_5	0.17	p_2	0.17
p_6	0	p_3	0

Lemma: 3 Let μ be a fuzzy subgroup on G defined by equation (1) and a$\in$ G then the fuzzy subgroup μ form a pseudo fuzzy coset.

Proof: Let μ be a fuzzy subgroup of a group G defined by the equation (1). Here the elements in the group $G = \{p_1, p_2, p_3, p_4, p_5, p_6\}$. Now let us choose the function $p(x)$ as follows.

$$p(x) = \mu(x^2) = \ 0.33\ \ if \quad x = p_1, p_2, p_3, p_4$$
$$= \quad 1 \quad if \quad x = p_5$$
$$= \ 0.83\ \ if \quad x = p_6$$

From the definition of p(x), the function value obtained by $(a\mu)^P(x)$ is in the interval [0,1]. For every elements x$\in$ G, the value of $p(x)\mu(x)$ is calculated as follows,

If $x = p_1$ then $p(x)\mu(x) = p(p_1)\mu(p_1) = 0.11.$

If $x = p_2$ then $p(x)\mu(x) = p(p_2)\mu(p_2) = 0.06.$

If $x = p_3$ then $p(x)\mu(x) = p(p_3)\mu(p_3) = 0.$

If $x = p_4$ then $p(x)\mu(x) = p(p_4)\mu(p_4) = 0.22.$

If $x=p_5$ then $p(x)\mu(x)=p(p_5)\mu(p_5)=0.83.$

If $x=p_6$ then $p(x)\mu(x)=p(p_6)\mu(p_6)=0.83.$

This calculation concludes, the value of $p(a)\mu(x)$, $\forall\ x\in G$ is in the interval [0,1] and also $(a\mu)^P(x)=p(a)\mu(x)$. Hence the fuzzy subgroup μ defined by the equation (1) forms a pseudo fuzzy coset.

Lemma: 4 Let μ be a fuzzy subgroup of a group G defined by the equation (1) then μ is not a fuzzy normal subgroup.

Proof: Let $G=\{p_1,p_2,p_3,p_4,p_5,p_6\}$ be a 3-degree symmetric group and μ is a fuzzy subgroup of a group G defined by the equation (1). Let us assume that μ is a fuzzy normal subgroup. By the definition of fuzzy normal subgroup $\mu(x)=\mu(y^{-1}xy)$ for all x,y$\in$G.

Let us choose $x=p_4,\ y=p_6$.

$\mu(p_4)=0.67,$

$p_6^{-1}p_4p_6=(1,\ 2,\ 3)\ (2,\ 3)\ (1,\ 3,\ 2)=p_2$.

$\mu(p_2)=0.17.$

$\mu(x)\neq\mu(y^{-1}xy).$

This is a contradiction. Hence our assumption is wrong. Hence the fuzzy subgroup of G defined by the equation (1) is not a fuzzy normal subgroup.

Theorem: 5 Let μ be a fuzzy subgroup of a symmetric group G which is the group of all permutations of {1, 2, 3} with group operation composition then μ is a conjugate fuzzy subgroup onto itself and also the number of conjugate fuzzy subgroups corresponding to μ on a symmetric group G is o(G).

Proof: Let μ be a fuzzy subgroup of a group G defined by the equation (1). By the definition of conjugate fuzzy subgroup, "Let λ and μ be two fuzzy subgroups of a group G. Then λ and μ are said to be conjugate fuzzy subgroups of G if for some g$\in$G, $\lambda(x)=\mu(g^{-1}xg)$ for every x$\in$G".

Let us choose $g=p_1=(1)$. The value of $g^{-1}=p_1=(1)$. The conjugate fuzzy subgroup corresponding to μ is computed with the following calculations.

$g^{-1}p_1g=(1)\ (1)\ (1)=p_1;$

$g^{-1}p_2g=(1)\ (1,\ 2)\ (1)=p_2;$

$g^{-1}p_3g=(1)\ (1,\ 3)\ (1)=p_3;$

$g^{-1}p_4g=(1)\ (2,\ 3)\ (1)=p_4;$

$g^{-1}p_5g=(1)\ (1,\ 2,\ 3)\ (1)=p_5;$

$g^{-1}p_6g=(1)\ (1,\ 3,\ 2)\ (1)=p_6;$

For every x$\in$G, the value of $g^{-1}xg$ is computed. Using these values the conjugate fuzzy subgroup is defined as follows,

$$\lambda_1(x)=\begin{cases}0.33 & if \quad x=p_1\\ 0.16 & if \quad x=p_2\\ 0 & if \quad x=p_3\\ 0.67 & if \quad x=p_4\\ 0.83 & if \quad x=p_5\\ 1 & if \quad x=p_6\end{cases} \qquad (2)$$

Now from the definition (2), the fuzzy subgroup λ_1 is same as the fuzzy subgroup μ. Hence the fuzzy subgroup μ is conjugate to itself.

Now let $g=p_6=(1,\ 3,\ 2)$. The value of $g^{-1}=p_5=(1,\ 2,\ 3)$. The conjugate fuzzy subgroup corresponding to μ is computed as follows,

$g^{-1}p_1g=(1,\ 2,\ 3)\ (1)\ (1,\ 3,\ 2)=p_1;$

$g^{-1}p_2g=(1,\ 2,\ 3)\ (1,\ 2)\ (1,\ 3,\ 2)=p_3;$

$g^{-1}p_3g=(1,\ 2,\ 3)\ (1,\ 3)\ (1,\ 3,\ 2)=p_4;$

$g^{-1}p_4g=(1,\ 2,\ 3)\ (2,\ 3)\ (1,\ 3,\ 2)=p_2;$

$g^{-1}p_5g=(1,\ 2,\ 3)\ (1,\ 2,\ 3)\ (1,\ 3,\ 2)=p_5;$

$g^{-1}p_6g=(1,\ 2,\ 3)\ (1,\ 3,\ 2)\ (1,\ 3,\ 2)=p_6;$

For every x$\in$G, the value of $g^{-1}xg$ is computed. Using these values the conjugate fuzzy subgroup is defined as follows,

$$\lambda_2(x)=\begin{cases}0.33 & if \quad x=p_1\\ 0 & if \quad x=p_2\\ 0.67 & if \quad x=p_3\\ 0.17 & if \quad x=p_4\\ 0.83 & if \quad x=p_5\\ 1 & if \quad x=p_6\end{cases} \qquad (3)$$

Now from (3), the fuzzy subgroup λ_2 and μ are said to be conjugate fuzzy subgroups of G.

Let $g=p_4=(2,\ 3)$. The value of $g^{-1}=p_4=(2,\ 3)$. On computation the conjugate fuzzy subgroup λ_3 is defined as follows,

$$\lambda_3(x) = \begin{cases} 0.33 & if \quad x = p_1 \\ 0 & if \quad x = p_2 \\ 0.17 & if \quad x = p_3 \\ 0.67 & if \quad x = p_4 \\ 1 & if \quad x = p_5 \\ 0.83 & if \quad x = p_6 \end{cases} \quad (4)$$

Now from the definition (4), the fuzzy subgroup λ_3 and μ are said to be conjugate fuzzy subgroups in G.

Let $g = p_2,\ p_3\ and\ p_5$ respectively then the conjugate fuzzy subgroups $\lambda_4, \lambda_5\ and\ \lambda_6$ are computed by the definition of conjugate fuzzy subgroup and listed as follows,

$$\lambda_4(x) = \begin{cases} 0.33 & if \quad x = p_1 \\ 0.17 & if \quad x = p_2 \\ 0.67 & if \quad x = p_3 \\ 0 & if \quad x = p_4 \\ 1 & if \quad x = p_5 \\ 0.83 & if \quad x = p_6 \end{cases} \quad (5)$$

$$\lambda_5(x) = \begin{cases} 0.33 & if \quad x = p_1 \\ 0.67 & if \quad x = p_2 \\ 0 & if \quad x = p_3 \\ 0.17 & if \quad x = p_4 \\ 1 & if \quad x = p_5 \\ 0.83 & if \quad x = p_6 \end{cases} \quad (6)$$

$$\lambda_6(x) = \begin{cases} 0.33 & if \quad x = p_1 \\ 0.67 & if \quad x = p_2 \\ 0.17 & if \quad x = p_3 \\ 0 & if \quad x = p_4 \\ 0.83 & if \quad x = p_5 \\ 1 & if \quad x = p_6 \end{cases} \quad (7)$$

Now the fuzzy subgroups λ_4 & μ, λ_5 & μ and also λ_6 & μ are conjugate fuzzy subgroups in G. In the 3-degree symmetric group the fuzzy set μ defined on equation (1) is conjugate to $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ and λ_6 respectively, which are defined on the equations (2) to (7). Hence the number of conjugate fuzzy subgroups corresponding to μ on a symmetric group G is o(G) = 6. Hence the theorem is proved.

Conclusion : In this research work, the new fuzzy subgroup is defined on the symmetric group of degree 3 with the fuzzy coset and pseudo fuzzy coset properties. Finally this paper concludes with a new fuzzy subgroup μ in G is a conjugate fuzzy subgroup onto itself and also the number of conjugate fuzzy subgroups corresponding to μ on a symmetric group G is o(G). My next task is to find some more properties of this fuzzy subgroup. I will try to find a generalized fuzzy subgroup for the symmetric group S_n, $\forall$ n and study the properties of generalized fuzzy subgroup and also try to find applications of fuzzy symmetric group in the various fields.

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