EIGENVALUES FOR ITERATIVE SYSTEMS OF NONLINEAR THIRD ORDER BOUNDARY VALUE PROBLEMS ON TIME SCALES

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Abstract: Values of $\lambda_1, \lambda_2, \ldots, \lambda_n$ are determined for which there exist positive solutions of the iterative system of dynamic equations, $u_i^{\Delta^3}(t) + \lambda_i\, a_i\, f_i\, (u_i\,(\sigma(t)) = 0$, $1 \le i \le n, u_{n+1}$ (t) = u_1(t), t∈[a,b] subject to the boundary conditions $\alpha_{11}u_i$(a) - $\alpha_{12}u_i$(b) = 0, $\alpha_{21}u_i^{\Delta}$(a) - $\alpha_{22}u_i^{\Delta}$(b) = 0,
$\alpha_{31}u_i^{\Delta^2}$(a) - $\alpha_{32}u_i^{\Delta^2}(\sigma$(b)) = 0, 1 ≤ i ≤ n. A Krasnosel'skii fixed point theorem is applied.

Keywords: Time scales, Boundary value problems, Positive solutions, Iterative system, Cone.

Introduction : The theory of time scales, which has received a lot of attention, was introduced by Stefan Hilger in order to unify the theory on continuous and discrete analysis. Their theory unifies the theories of differential equations and difference equations, and also it is able to extend these classical cases to cases in between, and can be applied on different types of time scales. A time scale T is an arbitrary closed subset of the real, and the cases when this time scale is equal to the real or to the integers that represent the classical theories of differential and of difference equations. Many other interesting time scales exist, and they give rise to plenty of applications in the study of population dynamic models. The book on the subject of time scales by Bohner and Peterson [5], summarizes and organizes much of the time scale calculus.

We consider the existence of positive solution to third order nonlinear differential equation on time scales,

$u_i^{\Delta^3}(t) + \lambda_i\, a_i\, f_i\, (u_i\,(\sigma(t)) = 0$, $1 \le i \le n$,

u_{n+1} (t) = u_1(t), t ∈[a,b] ⋯ (1.1)

$\alpha_{11}u_i$(a) - $\alpha_{12}u_i$(b) = 0,

$\alpha_{21}u_i^{\Delta}$(a) - $\alpha_{22}u_i^{\Delta}$(b) = 0,

$\alpha_{31}u_i^{\Delta^2}$(a) - $\alpha_{32}u_i^{\Delta^2}(\sigma$(b)) = 0, 1 ≤ i ≤ n. ⋯ (1.2)

We use the following notations for our convenience, $\gamma_i = \alpha_{i1} - \alpha_{i2}$, i=1, 2, 3, and β_j= aα_{j1} − bα_{j2}, j =1, 2. We make the following assumptions throughout:

(A_1) $f_i \in C([0,\infty],[0,\infty))$, 1 ≤ i ≤ n

(A_2) $\gamma_i < 0, i = 1,2,3.$

(A_3) a_i ∈C ([a, σ(b)], [0,∞)), 1 ≤ i ≤ n, and each does not vanish identically on any closed subinterval of [a, σ(b)],

(A_4) Each of $f_{i0} = lim_{x\to 0^+}\frac{f_i(x)}{x}$, $f_{i\infty} = lim_{x\to\infty}\frac{f_i(x)}{x}$, 1 ≤ i ≤ n exists as real positive numbers.

Greens Function And Bounds:

In this section, we estimate the bounds of the Greens function for the homogeneous two-point boundary value problem corresponding to (1.1) - (1.2) .

Theorem: The Greens function for the homogeneous problem -y^{Δ^3}=0, satisfying the boundary conditions (1.2), is given by

G(t, s) =

$$\begin{cases} \dfrac{1}{2\gamma_1\ \gamma_2\,\gamma_3}\begin{bmatrix} -\alpha_{12}\,\gamma_2\ \gamma_3\left(\sigma(b)-\sigma(s)\right)\left(\sigma(b)-\sigma^2(s)\right) \\ -\alpha_{22}\gamma_3(-\beta_1+t\gamma_1)\left(\sigma(b)+\sigma^2(b)-\sigma(s)-\sigma^2(s)\right) \\ -\alpha_{32}\left(p-t\gamma_1\left(\beta_2+\alpha_{21}\,\sigma(a)-\alpha_{22}\sigma(b)\right)+t^2\right)\gamma_1\ \gamma_2 \end{bmatrix} \\ \quad , a \le t \le s \le \sigma^3(b) \\ \dfrac{1}{2\gamma_1\ \gamma_2\,\gamma_3}\begin{bmatrix} -\alpha_{11}\gamma_2\,\gamma_3(\sigma(s)-a)(\sigma^2(s)-a) \\ +\alpha_{21}\gamma_3(-\beta_1+t\gamma_1)\left(\sigma(s)+\sigma^2(s)-a-\sigma(a)\right) \\ -\alpha_{31}\left(\left(p-t\gamma_1\left(\beta_2+\alpha_{21}\,\sigma(a)-\alpha_{22}\sigma(b)\right)+\ t^2\right)\gamma_1\ \gamma_2\right) \end{bmatrix} \\ \quad , a \le \sigma(s) \le t \le \sigma^3(b) \end{cases} \tag{2.1}$$

Where p = $\beta_1\left(\beta_2+\alpha_{21}\sigma(a)-\alpha_{22}\sigma(b)\right) - \gamma_2(a^2\alpha_{11}-b^2\alpha_{12})$

Theorem: Let G(t, s) be the Greens function for the homogeneous problem $-u^{\Delta^3}$=0, satisfying the boundary conditions (1.2). Then the inequality

$\gamma G(\sigma(s),s) \le G(t,s) \le G(\sigma(s),s)$, (2.2)

holds for all (t, s) ∈ $[a,\sigma^3(b)]\times[a,b]$,

$where\ 0 < \gamma = min\{m_1, m_2\} \le 1.$

Proof: The Greens function G(t, s) for the BVP (1.1) - (1.2) is given in equation (3.1).

Clearly, G(t, s) > 0, on $(a,\sigma^3(b))\times(a,b)$. (2.3)

For a ≤ t ≤ s ≤ $\sigma^3(b)$ *we have from* (A_2)

G(t, s) ≤ $G(\sigma(s), s)$ and for a ≤ $\sigma(s)$ ≤ t ≤ $\sigma^3(b)$
we have from (A_2) G(t, s) ≤ $G(\sigma(s), s)$.
Therefore, we have
G(t, s) ≤ $G(\sigma(s), s)$ for all (t, s) ∈ $[a, \sigma^3(b)] \times [a, b]$.
To establish, the other inequality, for a ≤ t ≤ s ≤ $\sigma^3(b)$ *we have from* (A_2)
G(t, s) ≥ $m_1 G(\sigma(s), s)$, for all(t, s) ∈ $[a, \sigma^3(b)] \times [a, b]$.
Where $m_1 = \frac{a_1}{b_1 + b_2 + b_3}$, and

$$a_1 = \frac{\alpha_{32}}{2\gamma_3}\left[\left(\frac{\beta_2 + \alpha_{21}\sigma(a) - \alpha_{22}\sigma(b)}{2\gamma_2}\right)^2 - \frac{p}{\gamma_1\gamma_2}\right],$$

$$b_1 = \frac{\alpha_{12}}{2\gamma_1}(\sigma(b) - a)^2,$$

$$b_2 = -\frac{\alpha_{22}}{\gamma_2}[(-\frac{\beta_1}{\gamma_1} + \sigma^3(b))(\sigma^2(b) - a)],$$

$$b_3 = -\frac{\alpha_{32}}{2\gamma_3}[[b - (\frac{\beta_2 + \alpha_{21}\sigma(a) - \alpha_{22}\,\sigma(b)}{2\gamma_2})]^2 - [\frac{\beta_2 + \alpha_{21}\sigma(a) - \alpha_{22}\sigma(b)}{2\gamma_2}]^2 + \frac{p}{\gamma_1\gamma_2}],$$

where $m_2 = \frac{c_1}{d_1 + d_2 + d_3}$, and

$$c_1 = \frac{\alpha_{31}}{2\gamma_3}\left[\left(\frac{\beta_2 + \alpha_{21}\sigma(a) - \alpha_{22}\sigma(b)}{2\gamma_2}\right)^2 - \frac{p}{\gamma_1\gamma_2}\right],$$

$$d_1 = \frac{\alpha_{11}}{2\gamma_1}(a - b)^2,$$

$$d_2 = -\frac{\alpha_{21}}{\gamma_2}[(- -\frac{\beta_1}{\gamma_1} + a)(\sigma^2(b) - a)],$$

$$d_3 = -\frac{\alpha_{31}}{2\gamma_3}[[b - (\frac{\beta_2 + \alpha_{21}\sigma(a) - \alpha_{22}\,\sigma(b)}{2\gamma_2})]^2 - [\frac{\beta_2 + \alpha_{21}\sigma(a) - \alpha_{22}\sigma(b)}{2\gamma_2}]^2 + \frac{p}{\gamma_1\gamma_2}].$$

Therefore we have
G(t, s) ≥ $\gamma\, G(\sigma(s), s)$, *for all* $(t, s) \in [a, \sigma^3(b)] \times [a, b]$, *where* $0 < \gamma = min\{m_1, m_2\} \leq 1$.
We note that an n-tuple $(u_1(t), u_2(t), \ldots\ldots u_n(t))$ is a solution of the eigen value problem (1.1) - (1.2) if and only if

$$u_i(t) = \lambda_i \int_a^{\sigma(b)} G(t, s)\, a_i\ (s) f_i(u_{i+1}(\sigma(s)))\, \Delta s,$$

$a \leq t \leq \sigma^3$(b), 1 ≤ i ≤ n,
And $u_{n+1}(t) = u_1(t)$, $a \leq t \leq \sigma^3(b)$, so that, in particular,

$$u_1(t) = \lambda_1 \int_a^{\sigma(b)} G(t, s_1)\, a_1\ (s_1) f_1(\lambda_2 \int_a^{\sigma(b)} G(\sigma(s_1), s_2)\, a_2\ (s_2) \times f_2(\lambda_3 \int_a^{\sigma(b)} G(\sigma(s_2), s_3)\, a_3\ (s_3) \ldots . \times f_{n-1}\left(\lambda_n \int_a^{\sigma(b)} G(\sigma(s_{n-1}), s_n)\, a_n(s_n) f_n\ \left(u_1\left(\sigma(s_n\ )\right)\right) \Delta s_n\right) \ldots\ldots \Delta s_3)\Delta s_2)\Delta s_1$$

Values of $\lambda_1, \lambda_2, \ldots\ldots \lambda_n$ for which there are positive solutions (positive with respect to a cone) of (1.1)-(1.2), will be determined via application of the following fixed point theorem [9].

Theorem: (Krasnosel'skii) Let B be a Banach space, and let $P \subset$B be a cone in B. Assume that Ω_1 and Ω_2 are open subsets of B with
o ∈ $\Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$, *and let* $T\colon P \cap (\overline{\Omega_2} \backslash \Omega_1) \to P$ *be a completely continuous operator such that either*
(i) $\|Tu\| \leq \|u\|, u \in P \cap \partial\Omega_1$, and $\|Tu\| \geq \|u\|, u \in P \cap \partial\Omega_2$, or
(ii) $\|Tu\| \geq \|u\|, u \in P \cap \partial\Omega_1$, and $\|Tu\| \leq \|u\|, u \in P \cap \partial\Omega_2$.
Then T has a fixed point in P ∩ $(\overline{\Omega_2} \backslash \Omega_1)$.

3. Positive Solution in a Cone:

In this section, we apply Theorem 2.3 to obtain solution in a cone (i.e, positive solution) of (1.1) - (1.2). For our construction,
let B=$\{x\colon [a, \sigma^3(b)] \to T\}$ with supermum norm
$\|X\|$ = sup{| $x(t)$ |: $t \in [a, \sigma^3(b)]$}
and define a cone $P \subset B$ *by*
$P = \{x \in B\colon x(t) \geq 0$
on $[a, \sigma^3(b)]$, *and* $min_{t \in [a, \sigma^3(b)]}\, x(t) \geq \gamma\|x\|\}$.
We next define an integral operator
T: P→ B, *for* $u \in P$, *by*

$$Tu\ (t) = \lambda_1 \int_a^{\sigma(b)} G(t, s_1)\, a_1\ (s_1)\, f_1\ (\lambda_2 \int_a^{\sigma(b)} G(\sigma(s_1), s_2)\, a_2\ (s_2) \times f_2(\lambda_3 \int_a^{\sigma(b)} G(\sigma(s_2), s_3)\, a_3\ (s_3) \ldots . \times f_{n-1}(\lambda_n \int_a^{\sigma(b)} G(\sigma(s_{n-1}), s_n)\, a_n(s_n) f_n\ (u_1(\sigma(s_n\ )))\Delta s_n) \ldots\ldots \Delta s_3)\Delta s_2)\Delta s_1. \quad (3.1)$$

Note from (A_1), (A_2), and (2.3) that, for u∈ P, Tu(t) ≥ 0 on [a, $\sigma^3(b)$].
Also, for u∈ P, *we have from* (2.2)*that*

$$Tu(t) \leq \lambda_1 \int_a^{\sigma(b)} G(\sigma(s_1), s_1)\, a_1(s_1)\ f_1(\lambda_2 \int_a^{\sigma(b)} G(\sigma(s_1), s_2)\, a_2\ (s_2) \times f_2(\lambda_3 \int_a^{\sigma(b)} G(\sigma(s_2), s_3)\, a_3\ (s_3) \ldots . \times f_{n-1}(\lambda_n \int_a^{\sigma(b)} G(\sigma(s_{n-1}), s_n) a_n(s_n) f_n\ (u_1(\sigma(s_n\ )))\Delta s_n) \ldots\ldots \Delta s_3)\Delta s_2)\Delta s_1.$$

So that

$$\|Tu(t)\| \leq \lambda_1 \int_a^{\sigma(b)} G(t, s_1)\, a_1\ (s_1) f_1(\lambda_2 \int_a^{\sigma(b)} G(\sigma(s_1), s_2)\, a_2\ (s_2) \times f_2\ (\lambda_3 \int_a^{\sigma(b)} G(\sigma(s_2), s_3)\, a_3\ (s_3) \ldots . \times f_{n-1}\ (\lambda_n \int_a^{\sigma(b)} G(\sigma(s_{n-1}), s_n)\ a_n(s_n) f_n\ (u_1(\sigma(s_n\ ))\ )\Delta s_n) \ldots\ldots \Delta s_3)\Delta s_2)\Delta s_1. \quad (3.2)$$

Next, if u∈ P, *we have from* (2.2), (3.1), *and* (3.2)*that*

$$\min_{t \in [a, \sigma^3(b)]} Tu(t) = \min_{t \in [a, \sigma^3(b)]} \lambda_1 \int_a^{\sigma(b)} G(t, s_1)\, a_1\ (s_1) f_1(\lambda_2 \int_a^{\sigma(b)} G(\sigma(s_1), s_2)\, a_2\ (s_2) \times f_2(\ldots . f_{n-1}\left(\lambda_n \int_a^{\sigma(b)} G(\sigma(s_{n-1}), s_n)\ a_n(s_n) f_n\ \left(u_1\left(\sigma(s_n\ )\right)\right) \Delta s_n\right) \ldots\ldots \Delta s_3)\Delta s_2)\Delta s_1.$$

$$\geq \lambda_1 \gamma \int_a^{\sigma(b)} G(\sigma(s_1), s_1)\, a_1\ (s_1) f_1(\lambda_2 \int_a^{\sigma(b)} G(\sigma(s_1), s_2)\, a_2\ (s_2) \times f_2(\ldots . \times f_{n-1}(\lambda_n \int_a^{\sigma(b)} G(\sigma(s_{n-1}), s_n)\ a_n(s_n) f_n\ (u_1(\sigma(s_n\ )))\Delta s_n) \ldots\ldots \Delta s_3)\Delta s_2)\Delta s_1 \geq \gamma\|Tu\|$$

Consequently, T: P → P.
In addition, the standard arguments shows that T is completely continuous.
From the above arguments, we know that the

$existence\ of\ positive\ solutions\ of\ (1.1)-(1.2)$ can be transferred to the existence of positive fixed points of the operator T. We seek suitable fixed points of T belonging to the cone P.

For our first result, define positive numbers L_1 and L_2, by

$$L_1 = max_{1\le i\le n}\left\{\left[\gamma\int_{\xi}^{\omega} G(\tau_i,s)a_i\ (s)\Delta s f_{i\infty}\right]^{-1}\right\}$$

$$L_2 = min_{1\le i\le n}\left\{\left[\int_a^{\sigma(b)} G(\sigma(s),s)a_i\ (s)\Delta s f_{i0}\right]^{-1}\right\}.$$

Theorem: Assume that conditions (A_1) - (A_4) are satisfied. Then, for each $\lambda_1,\lambda_2,\ldots\ldots,\lambda_n$ satisfying

$L_1 < \lambda_i < L_{2,}$ $1\le i\le n$ (3.3)

there exists an n-tuple $(u_1,u_2,\ldots\ldots u_n)$ satisfying (1.1),(1.2) such that $u_i(t) > 0$ on (a, $\sigma^3(b)$), $1\le i\le n$.

Proof: Let λ_j, $1\le j\le n$, be as in (3.3). And let $\varepsilon > 0$ $be\ cho$sen such that

$$max_{1\le i\le n}\left\{\left[\gamma\int_{\xi}^{\omega} G(\tau_i,s)a_i\ (s)\Delta s f_{i\infty}-\varepsilon\right]^{-1}\right\} \le min_{1\le j\le n}\lambda_{j,}$$

And $max_{1\le j\le n}\lambda_j \le \min\limits_{1\le i\le n}\left\{\left[\int_a^{\sigma(b)} G(\sigma(s),s)a_i\ (s)\Delta s f_{i0}+\epsilon\right]^{-1}\right\}.$

We seek fixed points of the completely continuous operator T: P→ P defined by (3.1).

Now, from the definitions of $f_{i0}, 1\le i\le n$, there exists an H_1>0 such that, for each $1\le i\le n$, $f_i(x)\le (f_{i0}+\epsilon)x, 0 < x\le H_1$.

Let u ∈ P with $\|u\| = H_1$. We first have from (2.2) and choice of ϵ, for a$\le s_{n-1}\le\sigma(b)$,

$$\lambda_n\int_a^{\sigma(b)} G(\sigma(s_{n-1}),\ s_n)a_n(s_n)f_n\left(u\left(\sigma(s_n)\right)\right)\Delta s_n$$
$$\le\ \lambda_n\int_a^{\sigma(b)} G(\sigma(s_n),\ s_n)a_n(s_n)f_n(u(\sigma(s_n)))\Delta s_n$$
$$< \int_a^{\sigma(b)} G(\sigma(s_n),\ s_n)a_n(s_n)(f_{n0}+\epsilon)u(\sigma(s_n))\Delta s_n$$
$$\le\ \lambda_n\int_a^{\sigma(b)} G(\sigma(s_n),\ s_n)a_n(s_n)\Delta s_n\ (f_{n0}+\epsilon)\|u\|$$
$$\le \|u\| = H_1.$$

It follows in a similar manner from (2.2) and choice of ϵ that, for a $\le s_{n-2}\le\sigma(b)$,

$$\lambda_{n-1}\int_a^{\sigma(b)} G(\sigma(s_{n-2}),s_{n-1})a_{n-1}(s_{n-1})\times$$
$$f_{n-1}\left(\lambda_n\int_a^{\sigma(b)} G(\sigma(s_{n-1}),s_n)a_n(s_n)f_n(u(\sigma(s_n)))\Delta s_n\right)\Delta s_{n-1}$$
$$\le\lambda_{n-1}\int_a^{\sigma(b)} G(\sigma(s_{n-1}),s_{n-1})a_{n-1}(s_{n-1})\Delta s_{n-1}\left(f_{n-1,0}+\epsilon\right)\|u\|$$
$$\le \|u\| = H_1$$

Continuing with this bootstrapping, we reach for a $\le$ t $\le\sigma^3(b)$,

$$\lambda_1\int_a^{\sigma(b)} G(t,s_1)a_1(s_1)f_1\left(\ldots.f_n\left(u\left(\sigma(s_n)\right)\right)\Delta s_n\ldots.\right)\Delta s_1\le H_1,$$

so that, for a $\le$ t $\le\sigma^3(b)$,

Tu(t) $\le H_1$, or $\|Tu\|\le H_1 = \|u\|$. If we set $\Omega_1 = \{x\in B:\ \|x\| < H_1\}$, then

$\|Tu\|\le\|u\|$, for u∈ $P\cap\partial\Omega_1$. (3.4)

Next, from the definitions of $f_{i\infty}$, $1\le i\le n$, there exists $\overline{H_2}$ > 0 such that, for each $1\le i\le n$,

$f_i(x)\ge(f_{i\infty}-\epsilon)x,\quad x\ge\overline{H_2}$.

Let $H_2 = max\left\{2H_1,\ \frac{\overline{H_2}}{\gamma}\right\}$. Let u∈ P and $\|u\| = H_2$.

Then $min_{t\in[a,\sigma^3(b)]}\,u(t)\ge\gamma\|u\|\ge\overline{H_2}$.

Consequently, from (2.2) and choice of ϵ, for a$\le s_{n-1}\le\sigma(b)$, we have that

$$\lambda_n\int_a^{\sigma(b)} G(\sigma(s_{n-1}),\ s_n)a_n(s_n)f_n\left(u\left(\sigma(s_n)\right)\right)\Delta s_n$$
$$\ge\ \lambda_n\int_a^{\sigma(b)} G(\sigma(s_{n-1}),\ s_n)a_n(s_n)f_n(u(\sigma(s_n)))\Delta s_n$$
$$\ge\lambda\int_a^{\sigma(b)} G(\tau_n,\ s_n)a_n(s_n)(f_{n\infty}-\epsilon)u(\sigma(s_n))\Delta s_n$$
$$\ge\gamma\,\lambda_n\int_a^{\sigma(b)} G(\tau_n,\ s_n)a_n(s_n)\Delta s_n\ (f_{n\infty}-\epsilon)\|u\|\ \ge \|u\| = H_2.$$

It follows similarly from (2.2) and choice of ϵ, for a $\le s_{n-2}\le\sigma(b)$,

$$\lambda_{n-1}\int_a^{\sigma(b)} G(\sigma(s_{n-2}),s_{n-1})a_{n-1}(s_{n-1})\times$$
$$f_{n-1}\left(\lambda_n\int_a^{\sigma(b)} G(\sigma(s_{n-1}),s_n)a_n(s_n)f_n(u(\sigma(s_n)))\Delta s_n\right)\Delta s_{n-1}$$
$$\ge\gamma\lambda_{n-1}\int_a^{\sigma(b)} G(\tau_{n-1},s_{n-1})a_{n-1}(s_{n-1})\Delta s_{n-1}\left(f_{n-1,\infty}-\epsilon\right)\|u\|$$
$$\ge\|u\| = H_2.$$

Again, using a bootstrapping, we reach

$$\mathrm{Tu}(\tau_1) = \lambda_1\int_a^{\sigma(b)} G(\tau_1,s_1)a_1(s_1)f_1\left(\ldots\ldots f_n\left(u\left(\sigma(s_n)\right)\right)\Delta s_n\ldots\right)\Delta s_1$$

$\ge\|u\| = H_2$,so that, $\|Tu\|\ge\|u\|$. So if we set $\Omega_2 = \{x\in B:\ \|x\| < H_2\}$, then $\|Tu\|\ge\|u\|$,

for u ∈ $P\cap\partial\Omega_{2.}$ (3.5)

Applying Theorem 2.3 to (3.4) and (3.5), we obtain that T has a fixed point u ϵ $P\cap(\overline{\Omega_2}\backslash\Omega_1)$.

As such, setting $u_1 = u_{n+1} = u$, we obtain a positive solution $(u_1,u_2,\ldots\ldots,u_n)$ of (1.1) - (1.2) given iteratively by

$$u_j(t) = \lambda_j\int_a^{\sigma(b)} G(t,s)a_j(s)f_j\left(u_{j+1}\left(\sigma(s)\right)\right)\Delta s,$$

$j = n, n-1,\ldots..1$

The proof is complete.

Prior to our next result, let ξ_i, $1\le i\le n$, be defined by

$$\int_a^{\sigma(b)} G(\sigma(b)_i,s)a_i(s)\Delta s = max_{t\in[a,\sigma^3(b)]}\int_a^{\sigma(b)} G(t,s)a_i(s)\Delta s$$

Then, we define positive numbers L_3 and L_4 by

$$L_3 = max_{1\le i\le n}\left\{\left[\gamma\int_a^{\sigma(b)} G(\tau_i,s)a_i\ (s)\Delta s f_{i0}\right]^{-1}\right\}$$

and

$$L_4 = max_{1\le i\le n}\left\{\left[\int_a^{\sigma(b)} G(\sigma(s),s)a_i\ (s)\Delta s f_{i\infty}\right]^{-1}\right\}.$$

Theorem: Assume that conditions $(A_1)-(A_4)$ are satisfied. Then, for each $\lambda_1,\lambda_2,\ldots.,\lambda_n$ satisfying

$L_3 < \lambda_i < L_4, 1\le i\le n$ (3.6)

there an n-tuple $(u_1(t),u_2(t),\ldots\ldots.u_n(t))$ satisfying (1.1)-(1.2) such that $u_i(t) > 0$ on (a, $\sigma^3(b)$), $1\le i\le n$.

Proof: Let $\lambda_j, 1\le j\le n$ be as in (3.6). And let $\epsilon > 0$ be chosen such that

$$max_{1\le i\le n}\left\{\left[\gamma\int_a^{\sigma(b)} G(\tau_i,s)a_i\ (s)\Delta s f_{i0-\epsilon}\right]^{-1}\right\}\le min_{1\le j\le n}\lambda_j,$$

and

$max_{1\le j\le n}\lambda_j \le max_{1\le i\le n}\left\{\left[\int_a^{\sigma(b)} G(\sigma(s),s)a_i\ (s)\Delta s f_{i\infty+\epsilon}\right]^{-1}\right\}$

Let T be the cone preserving, completely continuous operator that was defined by (3.1).

From the definition of $f_{i0,}$ 1≤ $i \le n$ there exists $\overline{H_3}$ > 0 such that, for each 1≤ $i \le n$,

$f_i(x) \ge (f_{i0} - \epsilon)x,\ 0 < x \le \overline{H_3}$.

Also, from the definitions of f_{i0}, it follows that $f_{i0}(0) = 0,\ 1 \le i \le n$, and so there exist 0 < $K_n < K_{n-1} < \cdots K_2 < \overline{H_3}$ such that

$\lambda_i f_i(t) \le \frac{\overline{H_3}}{\int_a^{\sigma(b)} G(\xi_i,s)a_i(s)\Delta s},\quad t \in [0, K_i],\ 3 \le i \le n,$ and

$\lambda_2 f_2(t) \le \frac{\overline{H_3}}{\int_a^{\sigma(b)} G(\xi_2,s)a_i(s)\Delta s},\ t \in [0, K_2]$. Choose $u \in P$ with $\|u\| = K_n$. Then, we have

$\lambda_n \int_a^{\sigma(b)} G(\sigma(s_{n-1}), s_n)a_n(s_n)f_n(u(\sigma(s_n)))\Delta s_n$

$\le \lambda_n \int_a^{\sigma(b)} G(\xi_n, s_n)a_n(s_n)f_n(u(\sigma(s_n)))\,\Delta s,$

$\le \frac{\int_a^{\sigma(b)} G(\xi_n,s_n)a_n(s_n)K_{n-1}\Delta s_n}{\int_a^{\sigma(b)} G(\xi_n,s_n)a_n(s_n)\Delta s_n} \le K_{n-1}.$

Bootstrapping yields the standard iterative pattern, and it follows that

$\lambda_2 \int_a^{\sigma(b)} G(\sigma(s_1), s_2)a_2(s_2)f_2(\ldots.)\,\Delta s_2 \le \overline{H_3}$. Then,

Tu(τ_1)=$\lambda_1 \int_a^{\sigma(b)} G(\tau_1, s_1)a_1(s_1)f_1(\lambda_2 \ldots.)\Delta s_1$

≥$\lambda_1\gamma \int_a^{\sigma(b)} G(\tau_1, s_1)a_1(s_1)(f_{1,0} - \epsilon)\|u\|\Delta s_1$≥$\|u\|$.

So, $\|Tu\| \ge \|u\|$. If we put $\Omega_1 = \{x \in B: \|x\| < K_n\}$, then, $\|Tu\| \ge \|u\|$, for $u \in P \cap \partial\Omega_1$. (3.7).

Since each $f_{i\infty}$ is assumed to be a positive real number, it follows that $f_i, 1 \le i \le n$, is unbounded at ∞. For each 1≤i≤n, set $f^*{}_i(x) = sup_{a\le s\le x} f_i(s)$. Then, it straight forward that, for each 1≤i≤n, $f^*{}_i$ is a nondecreasing real-valued function, $f_i \le f^*{}_i$, and

$lim_{x\to\infty}\frac{f^*{}_i(x)}{x}$=$f_{i\infty}$.

Next, by definition of $f_{i\infty}, 1 \le i \le n$, there exists $\overline{H_4}$ such that, for each 1≤i≤n, $f^*{}_i(x) \ge (f_{i\infty} + \epsilon)x$, x ≥ $\overline{H_4}$. It follows that there exists

$H_4 > \max\{\overline{2H_3}, \overline{H_4}\}$ such that, for each 1 ≤ i ≤ n, $f^*{}_i(x) \le f^*{}_i(H_4),\ 0 < x \le H_4$. Choose u ∈ P with$\|u\| = H_4$.

Then, using the usual bootstrapping argument, we have

Tu(t) = $\lambda_1 \int_a^{\sigma(b)} G(t, s_1)a_1(s_1)f_1(\lambda_2 \ldots..)\Delta s_1$

≤ $\lambda_1 \int_a^{\sigma(b)} G(t, s_1)a_1(s_1)f^*{}_1(\lambda_2 \ldots..)\Delta s_1$

≤ $\lambda_1 \int_a^{\sigma(b)} G(\xi_1, s_1)a_1(s_1)f^*{}_1(H_4)\Delta s_1$

≤ $\lambda_1 \int_a^{\sigma(b)} G(\xi_1, s_1)a_1(s_1)\Delta s_1\,(f_{i\infty} + \epsilon)H_4$≤$H_4$=$\|u\|$, and so $\|Tu\| \le \|u\|$. So, if we set

$\Omega_2 = \{x \in B: \|x\| < H_4\}$, then $\|Tu\| \le \|u\|$,

for u∈ $P \cap \partial\Omega_2$. (3.8)

Application of part (ii) of theorem 2.3 yields a fixed point u of T belonging to P∩ ($\overline{\Omega_2}\backslash\Omega_1$), which in turn, with $u_1 = u_{n+1} = u$, yields an-tuple $(u_1, u_2, \ldots\ldots, u_n)$ satisfying (1.1) - (1.2) for the chosen values of $\lambda_i, 1 \le i \le n$.

Conclusion:

In this paper, We determined the eigenvalue intervals for which there exist positive solutions of the iterative system of two-point nonlinear boundary value problem on time scales which unifies the result on continuous intervals and discrete intervals by using A Guo-Krasnosel'skii fixed point-theorem. These results are rapidly arising in the field of modeling and determination of flagellate protozoan in a viscous fluid in further research.

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