A NEW PARAMETRIC BERNSTEIN POLYNOMIAL

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Abstract: Some parametric generalization of the Bernstein polynomial is obtained here. Corresponding to that necessary and sufficient condition for uniform convergence is obtained.

Keywords: Bernstein Polynomial, Convergence.

Introduction :

Let $C[a,b]$be the space of continuous functions on$[a,b]$.If $F(x) \in C[a,b]$

$\| F \| = max_{a\le x\le b} |F(x)|$

Let $F(u) \in C[0,1]$; the Bernstein polynomial of order n is defined by [1], [3]

$$B_n(F(u),x) = \sum_{i=0}^{n} F\left(\frac{i}{n}\right) P_{n,i}(x),$$

$$P_{n,i}(x) = \binom{n}{i} x^i (1-x)^{n-i}$$

Let $t(\ge 0)$ be a real number.Then a parametric Bernstein operator can be defined as follows:

$$B_n^t\left(F,t,\frac{x}{t}\right) = \sum_{i=0}^{n}\sum_{j=0}^{t-1} F\left(\frac{i+j}{n+t-1}\right) P_{n,i}\left(\frac{x}{t}\right)$$

Let s_n be a sequence of natural numbers converging to t.

Let $n \in N$ where N is the set of natural numbers and let $F(u) \in C[0,1]$; the new parametric polynomials are defined by

$$C_n^t\left(F(u),s_{n,}\frac{x}{t}\right) = \frac{1}{s_n}\sum_{i=0}^{n}\left(\sum_{j=0}^{s_{n-1}} F\left(\frac{i+j}{n+2s_n-t-1}\right) P_{n,i}\left(\frac{x}{t}\right)\right)$$

If $s_n \to t$, then

$$C_n^t\left(F(u),s_{n,}\frac{x}{t}\right) \to tB_n^t\left(F,t,\frac{x}{t}\right)$$

and when $s_n \to t$ and $t = 1$ then

$$C_n^t\left(F(u),s_{n,}\frac{x}{t}\right) \to tB_n^t\left(F,t,\frac{x}{t}\right) \to B_n(F(u),x)$$

Thus $C_n^t\left(F(u),s_{n,}\frac{x}{t}\right)$ is an interesting parametric generalization of the Bernstein polynomials.

Some Results :

Lemma1. Here we prove the inequality

$$0 \le \max_{0\le x\le 1} C_n^t\left(\left(u-\frac{x}{t}\right)^2, s_n, \frac{x}{t}\right) < \frac{4}{3}\cdot\frac{(s_n-1)^2}{n^2} + \frac{13}{12}\cdot\frac{1}{n}$$

Proof. For $0 \le \frac{x}{t} \le 1$ we have

$\sum_{i=0}^{n} P_{n,i}\left(\frac{x}{t}\right) = 1$ .... (1)

$\sum_{i=0}^{n} iP_{n,i}\left(\frac{x}{t}\right) = n\left(\frac{x}{t}\right)$ .... (2)

$\sum_{i=0}^{n} i^2P_{n,i}\left(\frac{x}{t}\right) = n^2\frac{x^2}{t^2} - n\frac{x^2}{t^2} + n\frac{x}{t}$ .... (3)

$C_n^t\left(1,s_n,\frac{x}{t}\right) =$

$\frac{1}{s_n}\sum_{i=0}^{n}\left(\sum_{j=0}^{s_n-1} 1\right)P_{n,i}\left(\frac{x}{t}\right) = \sum_{i=0}^{n} P_{n,i}\left(\frac{x}{t}\right) = 1$

.... (4)

$$C_n^t\left(u,s_n,\frac{x}{t}\right) = \frac{1}{s_n}\sum_{i=0}^{n}\left(\sum_{j=0}^{s_n-1}\left(\frac{i+j}{n+2s_n-t-1}\right)\right)P_{n,i}\left(\frac{x}{t}\right)$$

$$= \frac{1}{n+2s_n-t-1}\left(\sum_{i=0}^{n} iP_{n,i}\left(\frac{x}{t}\right)\right) + \frac{s_n-1}{2(n+2s_n-t-1)}\left(\sum_{i=0}^{n} P_{n,i}\left(\frac{x}{t}\right)\right)$$

$= \frac{n\left(\frac{x}{t}\right)}{n+2s_n-t-1} + \frac{s_n-1}{2(n+2s_n-t-1)}$ .... (5)

$C_n^t\left(u^2,s_n,\frac{x}{t}\right) =$

$\frac{1}{s_n}\sum_{i=0}^{n}\left(\sum_{j=0}^{s_n-1}\left(\frac{i+j}{n+2s_n-t-1}\right)^2\right)P_{n,i}\left(\frac{x}{t}\right)\frac{1}{s_n(n+2s_n-t-1)^2}$

$$= \frac{1}{(n+2s_n-t-1)^2}\sum_{i=0}^{n} i^2P_{n,i}\left(\frac{x}{t}\right) + \frac{s_n-1}{(n+2s_n-t-1)^2}\sum_{i=0}^{n} iP_{n,i}\left(\frac{x}{t}\right) + \frac{(s_n-1)(2s_n-1)}{6(n+2s_n-t-1)^2}\sum_{i=0}^{n} P_{n,i}\left(\frac{x}{t}\right)$$

$= \frac{1}{(n+2s_n-t-1)^2}\left[n^2\frac{x^2}{t^2} - n\frac{x^2}{t^2} + n\frac{x}{t}\right] + \frac{s_n-1}{(n+2s_n-t-1)^2}n\left(\frac{x}{t}\right) + \frac{(s_n-1)(2s_n-1)}{6(n+2s_n-t-1)^2}$ .... (6)

From (5) we get

$$-2\frac{x}{t}C_n^t\left(u,s_n,\frac{x}{t}\right) = \frac{-2n\left(\frac{x}{t}\right)^2}{n+2s_n-t-1} + \frac{(s_n-1)\left(\frac{x}{t}\right)}{n+2s_n-t-1}$$

...... (7)

Since $C_n^t\left(F,s_n,\frac{x}{t}\right)$ is a sequence of positive linear operators, combining (4), (6) and (7) we get

$$0 \le C_n^t\left(\left(u-\frac{x}{t}\right)^2, s_n, \frac{x}{t}\right) = C_n^t\left(u^2,s_n,\frac{x}{t}\right) - 2\frac{x}{t}C_n^t\left(u,s_n,\frac{x}{t}\right) + \frac{x^2}{t^2}C_n^t\left(1,s_n,\frac{x}{t}\right)$$

$$= \left[\frac{n^2-n}{(n+2s_n-t-1)^2} - \frac{2n}{n+2s_n-t-1} + 1\right]\frac{x^2}{t^2}$$

$+[n/(n+2s\_n-t-1)\^2 \ +n(s\_n-1)/(n+2s\_n -t-1)\^2 \ -(s\_n-1)/(n+2s\_n -t-1)] \ x/t+(s\_n-1)(〖2s〗\_n -1)/(6(n+2s\_n-t-1)\^2)$

$= I_1 + I_2 + I_3$

$$I_1 = \frac{x^2}{t^2}\frac{1}{(n+2s_n-t-1)^2}[n^2-n-2n(n+2s_n-t-1)+(n+2s_n-t-1)^2]$$

$$= \frac{\left(\frac{x}{t}\right)^2}{(n+2s_n-t-1)^2}[(2s_n-t-1)^2-n] \leq \frac{(2s_n-t-1)^2\left(\frac{x}{t}\right)^2}{(n+2s_n-t-1)^2} \leq \frac{(2s_n-t-1)^2}{n^2} \leq \frac{(2s_n-1)^2}{n^2}$$

$$I_2 = \frac{\frac{x}{t}}{(n+2s_n-t-1)^2}[n+n(s_n-1)-(s_n-1)(n+2s_n-t-1)]$$

$$= \frac{\frac{x}{t}}{(n+2s_n-t-1)^2}[n-(2s_n-t-1)(s_n-1)]$$

$$= \frac{\frac{x}{t}}{(n+2s_n-t-1)^2}[n-(s_n-1)^2-(s_n-t)(s_n-1)]$$

$$\leq \frac{n\left(\frac{x}{t}\right)}{(n+2s_n-t-1)^2} \leq \frac{n\left(\frac{x}{t}\right)}{n^2} \leq \frac{1}{n},$$

$$I_3 = \frac{(s_n-1)(2(s_n-1)+1)}{6(n+2s_n-t-1)^2} \leq \frac{1}{3}\frac{(s_n-1)^2}{n^2} + \frac{1}{12n}\frac{2n(s_n-1)}{\left((n+2s_n-t-1)\right)^2} < \frac{(s_n-1)^2}{3n^2} + \frac{1}{12n}$$

Combining estimates for I_1, I_2, and I_3 we get

$$0 \leq C_n^t\left(\left(u-\frac{x}{t}\right)^2, s_n, \frac{x}{t}\right) \leq \frac{(2s_n-1)^2}{n^2} + \frac{1}{n} + \frac{(s_n-1)^2}{3n^2} + \frac{1}{12n}$$

$$= \frac{(2s_n-1)^2}{n^2} + \frac{(s_n-1)^2}{3n^2} + \frac{13}{12n} < \frac{4}{3}\frac{(s_n-1)^2}{n^2} + \frac{13}{12n}$$

Convergence Of Parametric Polinomial :

Theorem 1.It is necessary and sufficient that

$$\lim_{n\to\infty}\left(\frac{s_n}{n}\right) = 0$$

for every function $F(u) \in C[0,1]$,

$$lim_{n\to\infty} \|C_n^t\left(F, s_n, \frac{x}{t}\right) - F\left(\frac{x}{t}\right)\| = 0 \qquad \text{....... (8)}$$

Proof: Necessary Condition

In (8), letting $F(u) = u$, we get

$lim_{n\to\infty} \|C_n^t\left(u, s_n, \frac{x}{t}\right) - \frac{x}{t}\| = 0$ and $\lim_{n\to\infty} |C_n^t(u, s_n, 0) - 0| = 0.$

From (4) we get

$$C_n^t(u, s_n, 0) = \frac{s_n-1}{2(n+2s_n-t-1)}$$

hence

$$\lim_{n\to\infty}\frac{s_n-1}{(n+2s_n-t-1)} = 0,$$

$for\ \varepsilon = \frac{1}{2}$, there is a natural number K such that for $n \geq K$, we have

$$\frac{s_n-1}{(n+2s_n-t-1)} < \frac{1}{2}, s_n - 1 < n, 0 \leq \frac{s_n-1}{2n} < \frac{s_n-1}{(n+2s_n-t-1)};$$

we get

$$\lim_{n\to\infty}\left(\frac{s_n}{n}\right) = 0.$$

Sufficient Condition. From (3) we get

$$C_n^t\left(1, s_n, \frac{x}{t}\right) = 1 \qquad \left(0 \leq \frac{x}{t} \leq 1\right);$$

from Lemma 1 and $lim_{n\to\infty}\left(\frac{s_n}{n}\right) = 0$

we get

$$\lim_{n\to\infty}\max_{0\leq\frac{x}{t}\leq 1} C_n^t\left(\left(u-\frac{x}{t}\right)^2, s_n, \frac{x}{t}\right) = 0$$

By Korovkin's theorem [2] we get that, for every $F(u) \in C[0,1]$

$$\lim_{n\to\infty} \|C_n^t\left(F, s_n, \frac{x}{t}\right) - F\left(\frac{x}{t}\right)\| = 0.$$

Corollary 1. Let $F(u) \in C[0,1]$. Then

$$\lim_{n\to\infty} \|t.B_n^t\left(F, t, \frac{x}{t}\right) - F\left(\frac{x}{t}\right)\| = 0.$$

Proof: If $s_n \to t$ from theorem1 we get Corollary 1.

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