FUZZY COMPOSITE MODULAR SEQUENCE SPACE

ANINDITA BASU

Abstract : The author has made an attempt to fuzzify the functional analytic study of general class of composite classical sequence space $F(E_k, f)$ by defining a composite class $F_s(L(\mathbb{R}), f, p, s)$ of sequences of real valued fuzzy numbers using modulus function f, where $L(\mathbb{R})$ is a class of fuzzy numbers, $\{p_k\}$ is a bounded sequence of strictly +ve real numbers with inf $p_k > 0$ and s ≥ 0. Further, by defining a suitable topology on it several topological properties and inclusion relations between them have been investigated.

Keywords : Complete paranormed space, convergence free, Fuzzy numbers, modulus function, K-space, sequence algebra, solid space.

Introduction : The theory of fuzzy sets, is basically, a theory of graded concepts, that is, a theory in which everything is a matter of degree. In 1965, Zadeh [35] first introduced the concept and since then a large number of research papers have appeared by using the concept of fuzzy set/numbers and fuzzification of many classical theories has also been made.

Fuzzy set theory has become an important area of research in various branches of Mathematics such as Metric and topological spaces [7], Theory of functions [34], Approximation Theory [1] etc. Fuzzy set theory finds its applications in various fields of Science & Engineering, e.g., Computer Programming [11], Nonlinear Dynamical Systems [13], Population Dynamics[5], Control of Chaos [9], Quantum Physics [19] etc. It also attracted researchers on sequence spaces to introduce different type of classes of sequences of fuzzy numbers.

Puri and Ralescu[25] first introduced the concept of interval numbers by defining that if D be the set of all closed and bounded intervals of real numbers $X = [\bar{X}, \underline{X}]$, then the order relation ≤ may be defined on the set of intervals D as $X \leq Y \Leftrightarrow \bar{X} \leq \bar{Y}$. Also it is shown that (D, d) is a complete metric space, where

$$d(X, Y) = max\ \{\underline{X} - \underline{Y}, \bar{X} - \bar{Y}\}, \forall X, Y \in L(\mathbb{R})$$

Using this approach Matloka [20] introduced bounded and convergent sequences of fuzzy numbers where a fuzzy number X is a mapping $X: \mathbb{R} \longrightarrow [0,1]$ satisfying

(i) X is normal, i.e., ∃ an $t_0 \in \mathbb{R}$ such that $X(t_0) = 1$;

(ii) X is convex, i.e., $X(t) \geq X(s) \wedge X(r) = min\,(X(s), X(r))$, where $r < t < s$;

(iii) X is upper semicontinuous i.e., for each $\epsilon > 0$, $X^{-1}([0, a + \epsilon])$ is open in the usual topology of $\mathbb{R}$, for all $a \in [0, 1]$;

(iv) $X^0 = \{t \in \mathbb{R}: X(t) > 0\}$ is compact.

These properties imply that, for each $\alpha \in [0, 1]$, the α- cut of a fuzzy real number X,

$$X^\alpha = \{t \in \mathbb{R}: X(t) \geq \alpha\} = [\underline{X^\alpha}, \overline{X^\alpha}]$$

is a non-empty compact convex subset of $\mathbb{R}$. The class of fuzzy numbers satisfying the above conditions (i) and (ii) is denoted by $L(\mathbb{R})$. Matloka [20] defined a metric $\bar{d}: L(\mathbb{R}) \times L(\mathbb{R}) \longrightarrow \mathbb{R}$ on $L(\mathbb{R})$ by

$$\bar{d}(X, Y) = \sup_{0 \leq \alpha \leq 1} d(X^\alpha, Y^\alpha)$$

and proved that $(L(\mathbb{R}), \bar{d})$is a complete metric space. This metric $\bar{d}$ is translation invariant.

$\mathbb{R}$ can be embedded in $L(\mathbb{R})$, since each each $r \in \mathbb{R}$ can be regarded as a fuzzy number $\bar{r}$ defined by

$$\bar{r}(t) = \begin{cases} 1, & t = r \\ 0, & t \neq r \end{cases}$$

The additive identity and multiplicative identity of $L(\mathbb{R})$ are denoted by $\bar{0}$and $\bar{1}$ respectively.

An absolute value[14] of X∈ $L(\mathbb{R})$ is defined as

$$|X(t)| = \begin{cases} max\{X(t), X(-t)\}, & t \geq 0 \\ 0, & otherwise \end{cases}$$

Applying the notion of fuzzy real numbers, fuzzy real-valued sequence space has been introduced and studied by Nanda [22], at first. He showed that the set of all convergent sequences of fuzzy numbers forms a complete metric space. Later on the study of theory of fuzzy sequence spaces has been continued by Tripathy and Nanda[23], Savas [28], Mursaleen et al.[21], Fang et al.[8], Talo et al.[29], Tripathy et al.[31], Dutta et al.[12], Esi et al.[6] and many others.

On the other hand, the study of scalar and vector valued sequence spaces is one of the active areas of research in Functional analysis and it has been a subject of great interest to several mathematicians for the last several decades. It has vast application to different branches of analysis such as the theory of Banach spaces, topological vector spaces, duality theory, matrix transformation etc. In connection with the functional analytic study of theory of vector valued sequence spaces Ghosh and Srivastava[10] first introduced the concept of composite vector valued sequence space $F(E_k, f)$ where $E_k s$ are Banach spaces, f is a modulus function and F is a normal sequence space of real or complex numbers. Later on, Basu and Srivastava ([2], [4], [3]), Karakaya et al.[16] and many others extended this study for single and double sequences and their various convergence methods with the help of Orlicz function, modulus function, multiplier sequences etc.

Here, we have defined the composite modular class of fuzzy number sequences as follows:

The new class $F_s(L(\mathbb{R}), f, p, s)$

Let F_s be a normal sequence space (of real valued fuzzy numbers) with monotone paranorm g_{F_s} such that co-ordinatewise convergence implies convergence in paranorm g_{F_s} i.e.,

$X_k^n \longrightarrow 0$ as $n \longrightarrow \infty \Longrightarrow g_{F_s}(X^n) \longrightarrow 0$ as $n \longrightarrow \infty$ where $(X^n) = (X_k^n) \in F_s$. (1.1)

Let f be a modulus function. We define

$$F_s(L(\mathbb{R}), f, p, s) = \left\{X = (X_k): X_k \in L(R), \left(k^{-s}\left(f\left(\bar{d}(X_k, \overline{0})\right)\right)^{p_k}\right) \in F_s\right\}$$

where $\{p_k\}$ is a bounded sequence of strictly +ve real numbers with $\inf p_k > 0$ and $s \geq 0$.

As the paranorm g_{F_s} and the condition on g_{F_s} is irrelevant until and unless we define a topology on the composite fuzzy sequence space $F_s(L(\mathbb{R}), f, p, s)$, we define a topology on $F_s(L(\mathbb{R}), f, p, s)$ as follows:

For $X = (X_n) \in F_s(L(\mathbb{R}), f, p, s)$,

$$\overline{g}(X) = g_{F_s}\left[k^{-s}\left(f\left(\bar{d}(X_k, \overline{0})\right)\right)^{p_{k/M}}\right] \quad (1.2)$$

where $M = max(1,\ H)$, $H = sup_k\, p_k < \infty$, $inf\, p_k > 0$.

It is shown that this generalized class of sequences of fuzzy real numbers gives rise to many well known fuzzy sequence spaces on specifying the sequence space F_s, the modulus function f, the bounded sequence $\{p_k\}$ of positive real numbers and $s \geq 0$ as follows:

Let $c(F)$, $c_0(F)$, $l_\infty(F)$, ℓ_p(F) denote the fuzzy convergent, fuzzy null, fuzzy bounded, fuzzy p-th summable sequence spaces respectively.

Particular cases:

(i) If we take $F_s = c(F),\ \ell_p(F),\ s = 1, p_k = 1$, for each k, $f(x) = x$, $F_s(L(\mathbb{R}), f, p, s)$ gives rise the spaces of Nanda [22];

(ii) If we take $F_s = w(p,\ F),\ c_0(p,\ F),\ s = 1,\ f(x) = x$, $F_s(L(\mathbb{R}), f, p, s)$ gives rise the spaces of Mursaleen et al.[21];

(iii) If we take $F_s = \ell(p,\ F)$[20], $s = 1$, $f(x) = x$, $F_s(L(\mathbb{R}), f, p, s)$ gives rise the spaces of Nurray and Savas[24];

(iv) If we take $F_s = m(\varphi)$ [25], $s = 1$, $p_k = p$, $F_s(L(\mathbb{R}), f, p, s)$ gives rise space of Sarma[27];

(v) For $F_s = l_\infty(F), c(F),\ c_0(F),\ l_p(F),\ s = 1,\ p_k = 1$, $F_s(L(\mathbb{R}), f, p, s)$ gives rise space of Talo et al. [29].

We now quote the following definitions which will be needed in the sequel.

Some Definitions And Propositions:

Definition 1. [19] A sequence $X = (X_n)$ of fuzzy numbers is said to be convergent to the fuzzy number X_0 if for every $\varepsilon > 0$ there exists a positive integer n_0 such that

$$\bar{d}(X_n, X_0) < \varepsilon, \forall\, n > n_0$$

Definition 2. [19] A sequence $X = (X_n)$ of fuzzy numbers is said to be a Cauchy sequence if for every $\varepsilon > 0$ there exists a positive integer n_0 such that

$$\bar{d}(X_n, X_m) < \varepsilon, \forall\, n, m > n_0$$

Definition 3. (Paranorm)[17] Let X be a linear space. Then $g: X \longrightarrow \mathbb{R}$ is called a paranorm on X if for $x, y \in X$ and any scalar λ, (i) $g(x) \geq 0$; (ii) $x=\theta \Rightarrow g(x) = 0$; (iii) $g(x) = g(-x)$; (iv) $g(x + y) \leq g(x) + g(y)$; (v) $g(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$ whenever $\lambda_n \to \lambda$ and $x_n \to x$ for scalars λ_n, λ and $x_n, x \in X$.

Definition 4. (Monotone paranorm) [17] A paranorm g on a scalar valued paranormed sequence space Z is called monotone paranorm if

$X = (x_k) \in Z$, $Y = (y_k) \in Z$ and $|x_k| \leq |y_k|$ implies $g(x) \leq g(y)$.The space Z is called monotone paranormed space.

Definition 5. [15] By modulus function f, we mean a function $f: [0, \infty[\to [0, \infty[$ such that

(i) f(x) = 0 if and only if x = 0;

(ii) f(x + y) ≤ f(x) + f(y) for all x, y ≥ 0;

(iii) f is an increasing function;

(iv) f is continuous from the right at 0.

Definition 6. [25] A sequence $X = (X_n)$ of fuzzy real numbers is said to be bounded if the set $\{X_n: n \in \mathbb{N}\}$ is bounded. The space of bounded sequences of fuzzy numbers is denoted by $l_\infty(F)$.

Definition 7. [27] A sequence space λ of sequences fuzzy numbers is said to be solid (or normal) if $(Y_n) \in \lambda$, whenever $\bar{d}(Y_n, \overline{0}) \leq \bar{d}(X_n, \overline{0})$ for some $(X_n) \in \lambda$.

Definition 8. A sequence space λ of sequences fuzzy numbers is said to be symmetric if $\left(X_{\pi(n)}\right) \in \lambda$ whenever $(X_n) \in \lambda$, where π is a permutation on $\mathbb{N}$.

Definition 9. A sequence space λ of sequences fuzzy numbers is said to be sequence algebra if $(X_n Y_n) \in \lambda$ whenever $(X_n) \in \lambda$, $(Y_n) \in \lambda$.

Definition 10. A sequence space λ of sequences fuzzy numbers is said to be convergence free if $(X_n) \in \lambda$ when $(Y_n) \in \lambda$ and $Y_n = \overline{0}$ implies $X_n = \overline{0}$.

Definition 11. [21] A metric $\bar{d}$ is said to be translation invariant metric on $L(\mathbb{R})$ if $\bar{d}(X + Z, Y + Z) = \bar{d}(X, Y)$ for $X,\ Y,\ Z \in L(\mathbb{R})$.

Proposition 2.1 [18] Let f_1 and f_2 be two modulus functions and $0 < \delta < 1$. If $f_1(t) > \delta$ then

$$(f_2 \circ f_1)(t) \leq \frac{2 f_2\,(1)}{\delta} f_1(t)$$

holds for $\forall t \in [0, \infty[$.

Proposition 2.2 [29] Let $X, Y, Z, W \in L(\mathbb{R})$ and $k \in \mathbb{N}$. Then,(i) $\left(L(\mathbb{R}),\ \bar{d}\right)$ is a complete metric space [20];

(ii) $\bar{d}(kX,\ kY) = |k|\ \bar{d}(X, Y)$;

(iii) $\bar{d}(X + Y, W + Y) = \bar{d}(X, W)$;

(iv) $\bar{d}(X + Y, W + Z) \leq \bar{d}(X, W) + \bar{d}(Y, Z)$;

(v) $\left|\bar{d}(X,\ \bar{0}) - d(Y,\bar{0})\right| \le \bar{d}(X,Y) \le \bar{d}(X,\ \bar{0}) + \bar{d}(Y,\bar{0})$;

Proposition 2.3 [30] Let $X,Y,Z,W \in L(\mathbb{R})$ and $k \in \mathbb{N}$. Then

(i) $\bar{d}((XY,\bar{0}) \le \bar{d}((X,\bar{0})\, d(Y,\bar{0})$;

(ii)If $X_k \longrightarrow 0$ as $k \longrightarrow \infty$ then $\bar{d}((X_k,\ \bar{0}) \to \bar{d}((X,\ \bar{0})$as $k \longrightarrow \infty$;

Proposition 2.4 Let (p_k) be a bounded sequence of strictly positive real numbers with

$$0 < p_k \le \sup_k p_k = H, D = max(1, 2^{H-1}),$$
$$T = max(1, H)$$

Then

1. $|a_k + b_k|^{p_k} \le D\{|a_k|^{p_k} + |b_k|^{p_k}\}$;[18]
2. $|\lambda|^{p_k} \le max(1, |\lambda|^H)$;[17]
3. $[f(\lambda)]^{p_k}$<$(1, [|\lambda|^H])f(1)$; [17]

3. Main results.

Theorem 3.1 $F_s(L(\mathbb{R}), f, p, s)$ is a linear space.

Proof.Let $X = (X_k)$, $Y = (Y_k) \in F_s(L(\mathbb{R}), f, p, s)$ and $\lambda,\ \mu \in \mathbb{C}$. Then

$$k^{-s}\left(f\left(\bar{d}(\lambda X_k + \mu Y_k, \bar{0})\right)\right)^{p_k}$$
$$= k^{-s}\left(f\left(\bar{d}(\lambda X_k, \bar{0}) + \bar{d}(\mu Y_k, \bar{0})\right)\right)^{p_k}$$

(Proposition 2.4)

$$= k^{-s}\left(f\left(\begin{matrix}|\lambda|\bar{d}(X_k,\bar{0}) + \\ |\mu|\bar{d}(Y_k,\bar{0})\end{matrix}\right)\right)^{p_k}$$

(Proposition 2.2)

$$= D\{1 + [|\lambda|^H]\}\left(k^{-s}\left(f\left(\bar{d}(X_k,\bar{0})\right)\right)^{p_k}\right) + D\{1 + [|\mu|^H]\}\left(k^{-s}\left(f\left(\bar{d}(Y_k,\bar{0})\right)\right)^{p_k}\right)$$

(Proposition 2.5)

$\in F_s$, since F_s is normal.

So, $\lambda X + \mu Y \in F_s(L(\mathbb{R}), f, p, s)$.

Theorem 3.2 $F_s(L(\mathbb{R}), f, p, s)$ is a paranormed space under the paranorm given by 1.2.

Proof. Clearly $\bar{g}(\theta) = 0$, $\bar{g}(X) = \bar{g}(-X)$, where θ is the null element of $F_s(L(\mathbb{R}), f, p, s)$. Also by taking λ= μ = 1 in the previous theorem and using the fact that g_{F_s} is monotone, we get $\bar{g}(X + Y) \le \bar{g}(X) + \bar{g}(Y)$in $F_s(L(\mathbb{R}), f, p, s)$.

We are only left to show that $\bar{g}$ is continuous under scaler multiplication.

Suppose $(\lambda_m)$is a sequence of scalars such that $\lambda_m \to \lambda$ as $m \to \infty$. Let (X^m) be a sequence of elements of $F_s(L(\mathbb{R}), f, p, s)$ such that $X^m \overset{\bar{g}}{\to} X\ as\ m \to \infty i.e., \bar{g}(X^m - X) \to 0\ \ as\ m \to \infty$. To show $\bar{g}(\lambda_m X^m - \lambda X) \to 0\ \ as\ m \to \infty$.

We consider

$$\bar{g}(\lambda_m X^m - \lambda X)$$
$$= g_{F_s}\left[k^{-s}\left(f\left(\bar{d}(\lambda_m X_k^m - \lambda X_k, \bar{0})\right)\right)^{\frac{p_k}{M}}\right]$$
$$= g_{F_s}\left[k^{-s}\left(f\left(\bar{d}((\lambda_m - \lambda)(X_k^m - X_k) + \lambda(X_k^m - X_k) + (\lambda_m - \lambda)X_k, \bar{0})\right)\right)^{\frac{p_k}{M}}\right]$$
$$= g_{F_s}\left[k^{-s}\left(f\left(\bar{d}((\lambda_m - \lambda)(X_k^m - X_k), \bar{0})\right)\right)^{\frac{p_k}{M}}\right] + g_{F_s}\left[k^{-s}\left(f\left(\bar{d}(\lambda(X_k^m - X_k), \bar{0})\right)\right)^{\frac{p_k}{M}}\right] + g_{F_s}\left[k^{-s}\left(f\left(\bar{d}((\lambda_m - \lambda)X_k, \bar{0})\right)\right)^{\frac{p_k}{M}}\right]$$
$$= I_1 + I_2 + I_3 \text{ (say)}$$

As_$\lambda_m \to \lambda$ as $m \to \infty$ for suffciently large m, we can assume that $|\lambda_m - \lambda| < 1$. So,

$$I_1 = g_{F_s}\left[k^{-s}\left(f\left(|\lambda_m - \lambda|\bar{d}(X_k^m - X_k, \bar{0})\right)\right)^{\frac{p_k}{M}}\right] \quad \text{(By Proposition (2.2))}$$
$$< g_{F_s}\left[k^{-s}\left(f\left(\bar{d}(X_k^m - X_k, \bar{0})\right)\right)^{\frac{p_k}{M}}\right]$$

(since f is increasing)

$$= \bar{g}(X^m - X) \longrightarrow 0\ \ as\ m \to \infty \qquad (3.1)$$

Again,we have

$$I_2 = g_{F_s}\left[k^{-s}\left(f\left(|\lambda|\bar{d}(X_k^m - X_k, \bar{0})\right)\right)^{\frac{p_k}{M}}\right]$$ (By Proposition (2.2))

$$< (1 + [|\lambda|^H])g_{F_s}\left[k^{-s}\left(f\left(\bar{d}(X_k^m - X_k, \bar{0})\right)\right)^{\frac{p_k}{M}}\right]$$ (By Proposition (2.5))

$$= (1 + [|\lambda|^H])\,\bar{g}(X^m - X) \longrightarrow 0\ \ as\ m \to \infty \qquad (3.2)$$

Also, since $F_s$is normal and for suffciently large m, we can assume that$|\lambda_m - \lambda| < 1$,

$$k^{-s}\left(f\left(\bar{d}((\lambda_m - \lambda)X_k, \bar{0})\right)\right)^{\frac{p_k}{M}} < k^{-s}\left(f\left(\bar{d}(X_k, \bar{0})\right)\right)^{\frac{p_k}{M}} \in F_s,$$

since $X = (X_k) \in F_s(L(\mathbb{R}), f, p, s)$.

Again since $\lambda_m \to \lambda$ as $m \to \infty$, we have for each k,

$$k^{-s}\left(f\left(\bar{d}((\lambda_m - \lambda)X_k, \bar{0})\right)\right)^{\frac{p_k}{M}} \longrightarrow 0 \text{ as } m \to \infty,$$

Since g_{F_s} is monotone,

$$I_3 = g_{F_s}\left[k^{-s}\left(f\left(\bar{d}((\lambda_m - \lambda)X_k, \bar{0})\right)\right)^{\frac{p_k}{M}}\right] \longrightarrow 0 \text{ for}$$

sufficiently large m. Combining this with (3.1) and (3.2) we get the desired result.

Theorem 3.3$F_s(L(\mathbb{R}), f, p, s)$ is a K-space if F_s is a K-space.

Proof: Define $P_n: F_s(L(\mathbb{R}), f, p, s) \longrightarrow L(\mathbb{R})$ by $P_n(X) = X_n, \forall\, n \in \mathbb{N}$.

To show P_n is continuous.

Let (X^m) be a sequence in $F_s(L(\mathbb{R}), f, p, s)$ such that $X^m \overset{\bar{g}}{\to} 0$ as $m \longrightarrow \infty$.

Since F_s is a K-space, this implies for each k,

$$k^{-s}\left(f\left(\bar{d}(X_k^m - X_k, \bar{0})\right)\right)^{\frac{p_k}{M}} \longrightarrow 0,$$

as $m \to \infty$.

As f is continuous and increasing, $\bar{d}(X_k^m - X_k, \bar{0}) \longrightarrow 0$ as $m \to \infty$.

So, $X^m \longrightarrow \bar{0}\ as\ m \to \infty$ in $L(\mathbb{R})$.

Consequently, P_n is continuous and hence $F_s(L(\mathbb{R}), f, p, s)$ is a K-space.

Theorem 3.4 . $F_s(L(\mathbb{R}), f, p, s)$ is a normal space.
Proof: It follows trivially from the fact that f is increasing and F_s is normal.
Theorem 3.5 $F_s(L(\mathbb{R}), f, p, s)$ is a complete paranormed space with respect to $\bar{g}$ if F_s is normal K-space.
Proof: Let (X^m) be a Cauchy sequence in $F_s(L(\mathbb{R}), f, p, s)$.
Then $\bar{g}(X^m - X^n) \to 0$ $as\ m,\ n \to \infty$. Since f is increasing and F_s is a K-space,
$\bar{d}(X_k^m - X_k^n, \bar{0}) \longrightarrow 0$ $as\ m,\ n \to \infty$, for each k.
Thus (X_k^m) becomes a Cauchy sequence in $L(\mathbb{R})$,. Since $(L(\mathbb{R}),\ \bar{d})$is a complete metric space [19] there exist $X = (X_k) \in L(\mathbb{R})$such that $X^m \longrightarrow X$ as $m \to \infty$, for each k in $L(\mathbb{R})$, i.e., for each k, $\bar{d}(X_k^m - X_k, \bar{0}) \longrightarrow 0$ $as\ m \to \infty$, for each k.
Since f is continuous, it follows that for each k,
$\left(f\left(\bar{d}(X_k^m - X_k, \bar{0})\right)\right)^{p_k} \longrightarrow 0$ $as\ m \to \infty$, for each k.Then $k^{-s}\left(f\left(\bar{d}(X_k^m - X_k, \bar{0})\right)\right)^{p_k} \longrightarrow 0$ $as\ m \to \infty$, for each k.
Let $Y_k^m = k^{-s}\left(f\left(\bar{d}(X_k^m - X_k, \bar{0})\right)\right)^{p_k}$.
Then for each k, $Y_k^m \longrightarrow 0$ $as\ m \to \infty$, Then by suitable choice of δ (depending on m and k),

$$Y_k^m < \delta k^{-s}\left(f\left(\bar{d}(X_k^m, \bar{0})\right)\right)^{p_k}$$

where $0 < \delta < 1$, (3.3)
since f is increasing and $\bar{d}$ is translation invariant metric.
Since F_s is normal, it follows that $(Y^m) \in F_s$ for each m.
Moreover, as $Y_k^m \longrightarrow 0$ $as\ m \to \infty$ we have, $\bar{g}\,(Y^m) \longrightarrow 0$ as $m \to \infty$, Hence $X^m \overset{\bar{g}}{\to} 0$ as $m \longrightarrow \infty$ in $F_s(L(\mathbb{R}), f, p, s)$.
Again, $k^{-s}\left(f\left(\bar{d}(X_k, \bar{0})\right)\right)^{p_k}$

$$= k^{-s}\left(f\left(\bar{d}(X_k^m + X_k - X_k^m, \bar{0})\right)\right)^{p_k}$$
$$\leq k^{-s}\left(f\left(\bar{d}(X_k^m, \bar{0})\right)\right)^{p_k} + Y_k^m$$
$$\leq (1+\delta)k^{-s}\left(f\left(\bar{d}(X_k^m, \bar{0})\right)\right)^{p_k}$$

[by (3.3)]
Since $(X^m) \in F_s(L(\mathbb{R}), f, p, s)$ and by Theorem 3.4 , it follows that X=$(X_k) \in F_s(L(\mathbb{R}), f, p, s)$and hence $F_s(L(\mathbb{R}), f, p, s)$ is complete.

Maddox[18] proved that there is some modulus f such that $f(uv) = f(u)f(v)$ for $u \geq 0,\ v \geq 0$.

As example,$f(x) = log(1 + x)$.
Theorem 3.6 $F_s(L(\mathbb{R}), f, p, s)$is a sequence algebra if F_s is a normal sequence algebra.
Proof.Let $X = (X_k),\ Y = (Y_k) \in F_s(L(\mathbb{R}), f, p, s)$.
Consider

$$k^{-2s}\left(f\left(\bar{d}(X_kY_k, \bar{0})\right)\right)^{p_k}$$
$$= k^{-2s}\left(f\left(\bar{d}(X_k, \bar{0})\bar{d}(Y_k, \bar{0})\right)\right)^{p_k}$$

[Proposition 2.3]

$$= k^{-s}\left(f\left(\bar{d}(X_k, \bar{0})\right)\right)^{p_k} k^{-s}\left(f\left(\bar{d}(Y_k, \bar{0})\right)\right)^{p_k}$$

$\in F_s$, (by hypothesis)
since F_s is normal sequence algebra.
Consequently, $XY \in F_s(L(\mathbb{R}), f, p, s)$.
We give the definition of fuzzy composite Cesaro summable sequence as follows:
Definition 12 A sequence $X = (X_k)$ of fuzzy numbers is said to be fuzzy composite Cesaro summableto L if

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k\leq n}\left[k^{-s}\left(f\left(\bar{d}(X_k - \bar{L}, \bar{0})\right)\right)^{\frac{p_k}{M}}\right] = 0$$

Theorem 3.7 If a bounded sequence $X = (X_k)$ of fuzzy composite real numbers is convergent to L, then it is also fuzzy Cesaro summable to L.
Proof.Since $X \longrightarrow L$ in $l_\infty(F)(L(\mathbb{R}), f, p, s)$, for any given $\varepsilon > 0$, $\exists\ n_0 \in \mathbb{N}$ such that

$$\left[k^{-s}\left(f\left(\bar{d}(X_k - \bar{L}, \bar{0})\right)\right)^{\frac{p_k}{M}}\right] < \varepsilon,$$
$$\forall\ k \geq n_0.$$

Let $sup_k\left[k^{-s}\left(f\left(\bar{d}(X_k - \bar{L}, \bar{0})\right)\right)^{\frac{p_k}{M}}\right] < B$ for some $B > 0$.
Now,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k\leq n}\left[k^{-s}\left(f\left(\bar{d}(X_k - \bar{L}, \bar{0})\right)\right)^{\frac{p_k}{M}}\right]$$
$$\leq \lim_{n\to\infty} \frac{1}{n}\sum_{k\leq n_0}\left[k^{-s}\left(f\left(\bar{d}(X_k - \bar{L}, \bar{0})\right)\right)^{\frac{p_k}{M}}\right]$$
$$+ \lim_{n\to\infty} \frac{1}{n}\sum_{k=n_0+1}^{n}\left[k^{-s}\left(f\left(\bar{d}(X_k - \bar{L}, \bar{0})\right)\right)^{\frac{p_k}{M}}\right]$$

$=lim_{n\to\infty} \frac{1}{n}\{n_0 B + (n - n_0)\varepsilon\}$
=0.
But the converse is not necessarily always true.
Example 3.1. Consider the composite sequence of fuzzy numbers $X = (X_k)$, where F_s=$cs(F)$, the Cesaro summable space of fuzzy numbers, $f(x) = x,\ s = 1$, $p_k = 1$ and define

$$X_k = \begin{cases} \bar{\iota}, & i \in \mathbb{N}, k = i^2 \\ \bar{0}, & otherwise \end{cases}$$

Then the sequence $X = (X_k)$ is unbounded. But

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k\leq n}\bar{d}(X_k, \bar{0}) = \lim_{n\to\infty} \frac{1}{n^2}\frac{n(n+1)}{2}$$
$$= \lim_{n\to\infty} \frac{1}{2}(1 + \frac{1}{n})$$
$$= \frac{1}{2}$$

So $X = (X_k)$ is Cesaro summable to $\frac{1}{2}$.
Theorem 3.8 The composite sequence space $F_s(L(\mathbb{R}), f, p, s)$ is not always solid in general.
Proof.Let $F_s = l_\infty(F), c(F),\ c_0(\,F)$, then if $|\,X_k| \leq$

$|Y_k|$ for all $k \in \mathbb{N}$. Then
$\bar{d}(X_k, \bar{0}) \le \bar{d}(Y_k, \bar{0})$
Since f is increasing, it follows that

$$k^{-s}\left(f\left(\bar{d}(X_k, \bar{0})\right)\right)^{p_k} \le k^{-s}\left(f\left(\bar{d}(Y_k, \bar{0})\right)\right)^{p_k}$$

Again since $l_\infty(F), c(F), c_0(F)$ are solid [29], so $F_s(L(\mathbb{R}), f, p, s)$ is also solid.

We now consider the following examples motivated by the examples of double fuzzy sequences found in [32], [33]:

Example 3.2 For $F_s = l_\infty(F) \cap c(F)$, f(x) = x, $p_k = 1$, s = 1, consider the fuzzy real valued sequence(X_n)defined by

$$X_n(t) = \begin{cases} \frac{(2n+1)t-3}{2(n-1)}, & \frac{3}{2n+1} \le t \le 1 \\ \frac{(2n+1)t-3n}{1-n}, & 1 \le t \le \frac{3n}{2n+1} \\ 1, & otherwise \end{cases}$$

Then $lim_{n\to\infty} X_n = \bar{1}$.
So, $(X_n) \in l_\infty(F) \cap c(F)$.
Now, define (Y_n) by

$$Y_n = \begin{cases} \bar{1}, & n \ is \ odd \\ 2^{-1}, & otherwise \end{cases}$$

Then $|Y_n| \le |X_n|, \forall n \in \mathbb{N}$.
But $(Y_n) \notin l_\infty(F) \cap c(F)$.

Theorem 3.9 The composite sequence space $F_s(L(\mathbb{R}), f, p, s)$ is not always symmetric in general.

Proof. If we take $F_s = l_\infty(F) \cap c(F)$, f(x) = x, $p_k = 1$, s = 1, consider the fuzzy real valued sequence $(X_n) \in F_s$ satisfying $X_n \longrightarrow \bar{0}$ as $n \longrightarrow \infty$.
Then for any given $\varepsilon > 0$, $\exists\, n_0 \in \mathbb{N}$ such that

$$\bar{d}(X_n, \bar{0}) < \varepsilon, \forall n \ge n_0.$$

Let $Y_i = X_{n_i}, \ \forall i \in \mathbb{N}$
and $R = max\{n_1, n_2, \ldots, n_{n_0}\}$.
Then for all $n > R$, we have $\bar{d}(Y_n, \bar{0}) < \varepsilon$ and hence $(X_n) \in l_\infty(F) \cap c(F)$.
Hence it is symmetric.
But consider the (X_n),

$$X_1(t) = \begin{cases} \frac{2+(1+t)n}{2+n}, & -(1+\frac{2}{n}) \le t \le 0 \\ \frac{2+(1-t)n}{2+n}, & 0 \le t \le \left(1+\frac{2}{n}\right) \\ 0, & otherwise \end{cases}$$

And $X_n = \bar{0}$ for $n \ne 1$.
Then $(X_n) \in c(F), \ c_0(F), \ l_\infty(F) \cap c_0(F)$, $l_\infty(F) \cap c(F)$,
Now consider the sequence(Y_n) define by

$$Y_n(t) = \begin{cases} \frac{2+(1+t)n}{2+n}, & -(1+\frac{2}{n}) \le t \le 0 \\ \frac{2+(1-t)n}{2+n}, & 0 \le t \le \left(1+\frac{2}{n}\right) \\ 0, & otherwise \end{cases}$$

And $Y_1 = \bar{0}$.
Then (Y_n) is a re-arrangement of (X_n) and $(Y_n) \notin c(F), \ c_0(F), \ l_\infty(F) \cap c_0(F)$, $l_\infty(F) \cap c(F)$,
So these spaces are not symmetric.

Theorem 3.10 The composite sequence space $F_s(L(\mathbb{R}), f, p, s)$ is not always convergence free in general.

Proof.
If we take $F_s = c(F), \ c_0(F), \ l_\infty(F) \cap c(F)$, f(x)=x, $p_k = 1$, s=1, and the fuzzy sequence (X_n) defined by

$$X_n(t) = \begin{cases} (1+2nt), & -\frac{1}{2n} \le t \le 0 \\ (1-2nt), & 0 \le t \le \frac{1}{2n} \\ 0, & otherwise \end{cases}$$

Then $lim_{n\to\infty} X_n = \bar{1}$.
So, $(X_n) \in c(F), \ c_0(F), \ l_\infty(F) \cap c(F)$.
But, if we consider the fuzzy sequence (Y_n) defined by

$$Y_n(t) = \begin{cases} (1+\frac{t}{2n}), & -2n \le t \le 0 \\ (1-\frac{t}{2n}), & 0 \le t \le 2n \\ 0, & otherwise \end{cases}$$

Then$(Y_n) \notin c(F), \ c_0(F), l_\infty(F), \ l_\infty(F) \cap c(F)$. These spaces are not convergence free.

Theorem 3.11 Let F_s be a normal space. Then the following inclusions hold:

1. If $k^{-s} \in F_s$ and $(f_2 \circ f_1)(t) = f_2(f_1(t), \ H = sup_k p_k < \infty, \ inf \ p_k > 0$, then
$$F_s(L(\mathbb{R}), f_1, p, s) \subseteq F_s(L(\mathbb{R}), f_2 \circ f_1, p, s);$$
2. $F_s(L(\mathbb{R}), f_1, p, s) \cap F_s(L(\mathbb{R}), f_2, p, s) \subseteq (L(\mathbb{R}), f_1 + f_2, p, s)$;
3. If $lim\, sup_{t>0} \frac{f_1(t)}{f_2(t)} < \infty, \ F_s(L(\mathbb{R}), f_2, p, s) \subseteq F_s(L(\mathbb{R}), f_1, p, s)$;
4. If $s_1 \le s_2$ then $F_s(L(\mathbb{R}), f_2, p, s_1) \subseteq F_s(L(\mathbb{R}), f_1, p, s_2)$;

Proof.(1) Let us choose δ such that $0 < \delta < 1$. Let

$$N_1 = \left\{k \in \mathbb{N}: f_1\left(\bar{d}(X_k, \bar{0})\right) \le \delta\right\}$$
$$N_2 = \left\{k \in \mathbb{N}: f_1\left(\bar{d}(X_k, \bar{0})\right) > \delta\right\}$$

Now,$k \in N_1 \Rightarrow (f_2 \circ f_1)\left(\bar{d}(X_k, \bar{0})\right) \le f_2(\delta)$ (since f is increasing)

So,$k^{-s}\left((f_2 \circ f_1)\left(\bar{d}(X_k, \bar{0})\right)\right)^{p_k} \le k^{-s} D_1$,
where $D_1 = [1 + |\delta|^H] f_2(1), \ H = sup\, p_k$.
For $k \in N_2$,

$$k^{-s}\left((f_2 \circ f_1)\left(\bar{d}(X_k, \bar{0})\right)\right)^{p_k}$$
$$\le k^{-s}\left(\frac{2f_2(1)}{\delta} f_1\left(\bar{d}(X_k, \bar{0})\right)\right)^{p_k}$$
$$\le D_2 k^{-s}\left(f_1\left(\bar{d}(X_k, \bar{0})\right)\right)^{p_k}$$

where $D_2 = max\left(1, \left[\left|\frac{2f_2(1)}{\delta}\right|^H\right]\right)$. Let $D = max(D_1, D_2)$.
Therefore, for $k \in N_1 \cup N_2$,

$$k^{-s} \le D\left[k^{-s} + k^{-s}\left(f_1\left(\bar{d}(X_k, \bar{0})\right)\right)^{p_k}\right]$$

Since $k^{-s} \in F_s$ and $F_s(L(\mathbb{R}), f, p, s)$ is normal, the result follows.

Proof (2). The proof follows from the fact that F_s is normal and

$$\left(f_1\left(\bar{d}(X_k,\bar{0})\right)+\,f_2\left(\bar{d}(X_k,\bar{0})\right)\right)^{p_k}$$

$$\leq D\left[\left(f_1\left(\bar{d}(X_k,\bar{0})\right)\right)^{p_k}+\left(f_2\left(\bar{d}(X_k,\bar{0})\right)\right)^{p_k}\right]$$

[Proposition 2.5]

where $D=max(1,2^{H-1})\,, H=sup_k\,p_k$.

Proof (3).

$lim\,sup_{t>0}\frac{f_1(t)}{f_2(t)}<\infty\Rightarrow\exists$ some $L\ >0$ such that $f_1(t)\leq L\,f_2(t),\,\forall t>0$.

Now the proof follows from the fact that F_s is normal and

$$\left(f_1\left(\bar{d}(X_k,\bar{0})\right)\right)^{p_k}\leq\left(Lf_2\left(\bar{d}(X_k,\bar{0})\right)\right)^{p_k}$$

where$=sup_k\,p_k$..

Proof 4. Let $s_1\leq s_2$. Then $k^{-s_2}\leq k^{-s_1}$, since $0\leq k^{-1}\leq 1$.

Thus for each k,

$$k^{-s_2}\left(f\left(\bar{d}(X_k,\bar{0})\right)\right)^{p_k}\leq k^{-s_1}\left(f\left(\bar{d}(X_k,\bar{0})\right)\right)^{p_k}$$

Since F_s is normal, the result follows.

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Assistant Professor, Department of Mathematics, Dr. B. N. Dutta Smriti Mahavidyalaya, Burdwan-713407, West Bengal, India.
e.mail: anin_iitkgp@yahoo.com