THE EFFECT OF WALL ABSORPTION ON DISPERSION IN AN ANNULAR FLOW OF COUPLE STRESS FLUID

INDIRA R RAO, JAYAPRAKASH M C

Abstract : The unsteady convective diffusive mass transfer in a flow of couple- stress fluid in a concentric annulus is considered. The velocity is obtained for a couple stress fluids analytically solving fourth order equation, using no-slip and vanishing of couple stresses at the boundaries. The species equation is solved using the generalized dispersion model developed by Gill and Sankara Subramanian. The exchange coefficient M_0, convective coefficient-M_1 and dispersion coefficient M_2 are calculated and graphically depicted for different values of absorption parameter 'β' and couple stress parameter 'α'. Couple stresses parameter is inversely proportional to the spin of suspended particles the dispersion and convective coefficient decreases with increase in couple stress parameter where as absorption coefficient is not affected. This indicates angular velocity of the suspended particles help in enhancing dispersion and there by more solute is transported.

Keywords : couple stress fluid, generalized dispersion, exchange coefficient, convective coefficient, dispersion coefficient.

Introduction: The dispersion of tracer in a tubular/annular flow when it is either irreversibly absorbed or when it undergoes an exchange process at the boundary has many applications. The efficiency of mixing within the flow channel is vital in determining the loss of performances.

The problem of heat and mass transfer in an annulus bounded by two horizontal cylinders has been a subject of great interest due to its wide technological applications like nuclear reactors, geothermal recovery, exothermic chemical reaction etc (Tsui and Tremblay[1]). An example from clinical medicine is the dye or indicator dilution technique which is a common practice among the physiologists. Catheters are used to inject the dye and to withdraw blood samples for the purpose of measurements. Study of blood – tissue exchange in the physiological system is done by multiple indicator dilution technique. The dispersion of mass or heat in blood vessels is influenced by the conductive blood vessel walls as well as inserted catheter. Modeling of this situation is usually done by considering the flow as the annular flow between the inserted catheter and the blood vessel wall (Sarkar and Jayaraman [2],[3], Dash et.al [4], Back et.al [4]).

Sankarasubramanian and Gill [5] have analyzed the dispersion of solute undergoing first order wall reaction in Poiseuille flow through a circular tube. Their generalized dispersion model gave rise to three effective coefficients, namely convection, diffusion and exchange coefficients. Their approach was further developed by De Gance and John [6] to the case of a cylinder with arbitrary cross-section.

Dash [7] has studied the effect of wall conductance on the axial dispersion in the pulsatile flow. Jiang and Grotberg [8] have studied the dispersion of a bolus contaminant in a straight tube with oscillatory field and weak conductive walls. Smith and Walton [9] have studied the dispersion of solutes in an inclined flow in an annulus. Sarkar and Jayaraman [10] have studied effect of permeable wall characteristics in the study of dispersion of solute in an annulus.

In the present study effect of permeable wall and effect of couple stress in the fluid flow on dispersion in a concentric annulus is studied.

Mathematical Formulation : Fig. 1 shows a schematic diagram of the concentric annular geometry with an outer tube as an artery of radius R_0 and an inner tube of radius KR_0 ($K < 1$). The governing equation for steady flow of a couple stress fluid in an annulus is

$$\frac{\partial p}{\partial r}+\mu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)-\eta\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)\right)^2 w=0 \quad (1)$$

Using non dimensionalising condition $r^* = r/R_0$ and $w^* = w/w_0$ the result obtained is

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)\right]^2 w - a_0^2\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)=-P \quad (2)$$

where $a_0=\sqrt{\frac{\mu}{2R_0^2}}$, $P=-\frac{a_0^2}{R_0^2}\frac{\partial p}{\partial x}$. The above equation is subjected to no- slip at the boundary condition and it eliminates the couple stress. i.e

$$w=0,\ \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)=0 \text{ at } r=k \text{ and } r=1 \quad (3)$$

solving the equation (2) and using boundary conditions (3) the result obtained is

$$w=-\frac{P}{a_0}\left\{AI_0(a_0r)+BK_0(a_0r)-\frac{r^2}{2}-\frac{c}{r}-D\right\} \quad (4)$$

The constants are listed in the appendix.

The convective diffusion equation for dispersion in an annular flow of a solute with concentration c is given by
$$\frac{\partial c}{\partial t}+w(r,t)\frac{\partial c}{\partial z}=D\left\{\frac{\partial^2 c}{\partial r^2}+\frac{1}{r}\frac{\partial c}{\partial r}+\frac{\partial^2 c}{\partial z^2}\right\} \quad (5)$$

Initial And Boundary Conditions : The initial distribution at τ = 0 is considered as the case when the solute of mass 'm' is introduced instantaneously at the plane z = 0 uniformly over the section of an annulus k < r ≤ a. in dimensionless quantities the initial distribution is given by
$$c(0,r,\theta,z)=B_2(z)B_1(r) \quad (6)$$
where $B_2(z)=\frac{\delta(z)}{\alpha^2 Pe^2}$ and $B_1(r)=\begin{cases}1 & k<r\le\alpha\\ 0 & \alpha<r\le 1\end{cases}$ $\delta(z)$ is the Dirac delta function. Boundary conditions assuming reaction mechanism at the wall are given by
$$\frac{\partial c}{\partial r}=\begin{cases}-\beta c & \text{at } r=1\\ 0 & \text{at } r=k\end{cases} \quad (7)$$

4. Method of solution: In order to solve the convection- diffusion equation (5) along with the associated sets of initial conditions (6), and boundary conditions (7), the concentration is expressed as
$$c(\tau \quad r \quad z)=\sum_{n=0}^{\infty} f_n(\tau,r)\frac{\partial^n c_m}{\partial z^n} \quad (8)$$
where the average concentration 'c_m' is expressed as
$$c_m=\frac{\int_0^{2\pi}\int_k^1 rc\,dr\,d\theta}{\int_0^{2\pi}\int_k^1 r\,dr\,d\theta}=\frac{2}{(1-k^2)}\int_k^1 rc\,dr \quad (9)$$
and the f_n's have to be determined using (6), (7) and (8). It's assumed that the distribution of mean concentration 'c_m' can be described by the generalized dispersion model truncated after three terms as
$$\frac{\partial c_m}{\partial \tau}=M_0(\tau)c_m+M_1(\tau)\frac{\partial c_m}{\partial z}+M_2(\tau)\frac{\partial^2 c_m}{\partial z^2} \quad (10)$$
The term $M_0(\tau)$ corresponds to the absorption parameter. This term arises because of the nonzero solute flux at the flow boundary $M_0(\tau)$ will be zero if there were no absorption in the wall. $M_1(\tau)$ and $M_2(\tau)$ correspond to the convective and dispersion coefficients, respectively. The convection coefficient $M_1(\tau)$ accounts for the velocity of the reactive tracer, and the dispersion coefficient $M_2(\tau)$ provides the modifications in the convective dispersion occurring because of absorption.

Substituting (8) in (5) and using (10) to evaluate $\frac{\partial c_m}{\partial \tau}$ and expressing $\frac{\partial^{n+1} c_m}{\partial \tau \partial z^n}$ in terms of $\frac{\partial^n c_m}{\partial z^n}$ and equating the coefficients of $\frac{\partial^n c_m}{\partial z^n}$, the set of differential equations for f_n is
$$\frac{\partial f_n}{\partial \tau}=\frac{\partial^2 f_n}{\partial r^2}+\frac{1}{r}\frac{\partial f_n}{\partial r}-w(\tau,r)f_{n-1}+\frac{1}{Pe^2}f_{n-2}-\sum_{i=0}^{n} f_{n-i}M_i \quad (11)$$
where $f_{-1}=0=f_{-2}$. From the expression of c_m in (9) and using equations (6) and (7) we have
$$c_m(0,z)=\frac{2B_2(z)}{(1-k^2)}\int_k^1 r\,B_1(r)\,dr \quad (12)$$
Setting $f_n(0,r)$ = 0 for n = 1,2,3,... and using (8) we get
$$c(0,r,z)=f_0(0,r)c_m(0,z) \quad (13)$$
From (13), the initial condition on f_n is obtained using the expression of $c(0,r,z)$ from (13) and $c_m(0,z)$ from (12) as
$$f_n(0,r)=\begin{cases}\dfrac{(1-k^2)B_1(r)}{2\int_k^1 rB_1(r)dr} & n=0\\ 0 & n\neq 0\end{cases} \quad (14)$$
The boundary conditions are given by
$$\frac{\partial f_n(\tau,1)}{\partial r}=-\beta f_n(\tau,1) \quad \text{and} \quad \frac{\partial f_n(\tau,k)}{\partial r}=0 \quad (15)$$
Substituting the expression for c_m from (8) in (9) we get
$$\int_k^1 rf_n(\tau,r)dr=\frac{(1-k^2)}{2}\delta_{n,0} \quad (16)$$

The function f_0 and exchange coefficient M_0 are independent of the velocity field and can be solved directly. From (11) we have
$$M_0(\tau)=\frac{2}{(1-k^2)}\frac{\partial f_0}{\partial r}(\tau,1) \quad (17)$$
Thus, $f_0(\tau,r)$ has to be evaluated first to determine $M_0(\tau)$.

The coefficient $M_0(\tau)$ is first decoupled from the equation obtained from (10),
$$\frac{\partial f_0}{\partial \tau}=\frac{\partial^2 f_0}{\partial r^2}+\frac{1}{r}\frac{\partial f_0}{\partial r}-M_0 f_0 \quad (18)$$
by introducing the transformation $f_0(\tau,r)=e^{-\int_\tau^{\gamma 1} M_0(\eta)d\eta}g_0(\tau,r)$ we get
$$\frac{\partial g_0}{\partial \tau}=\frac{\partial^2 g_0}{\partial r^2}+\frac{1}{r}\frac{\partial g_0}{\partial r} \quad (19)$$
with the initial and boundary conditions as
$$g_0(0,r)=\frac{(1-k^2)B_1(r)}{\int_k^1 rB_1(r)dr}, \qquad \frac{\partial g_0}{\partial r}(\tau,k)=0$$
$$\frac{\partial g_0}{\partial r}(\tau,1)=-\beta g_0(\tau,1) \quad (20)$$

The solution of g_0 satisfying the initial and boundary conditions (4.17) – (4.19) is given by

$$g_0(\tau,r)=\sum_{n=0}^{\infty}\frac{A_n}{J_1(\mu_n k)}E_n(\mu_n r)e^{-\mu_n^2 r} \quad (21)$$

where
$$A_n=\frac{\mu_n^2\left(1-k^2\right)J_1(\mu_n k)\int_k^1 rB_1(r)E_n(\mu_n r)dr}{\left(\mu_n^2+\beta^2-k^2\mu_n^2\right)\{E_n(\mu_n)\}^2\int_k^1 rB_1(r)dr} \quad (22)$$

μ_n's are the Eigen values satisfying the equation, $\Psi(\mu_n)=0$

$$\Psi(\mu_n)=\mu_n\left[Y_1(\mu_n k)J_1(\mu_n)-J_1(\mu_n k)Y_1(\mu_n)\right]+\beta\left[Y_0(\mu_n)J_1(\mu_n k)-Y_1(\mu_n k)J_0(\mu_n)\right] \quad (23)$$

The Eigen functions $E_n(\mu_n r)$ are given by $E_n(\mu_n r)=Y_0(\mu_n r)J_1(\mu_n k)-Y_1(\mu_n r)J_0(\mu_n k)$ (2)

Using the transformation and equation (16) we get,

$$e^{-\int_0^{\gamma} M_0(\tau)d\tau}=\frac{\left(1-k^2\right)}{2\int_k^1 r g_0(\tau,r)dr}. \quad (24)$$

From (17) and (24) we have

$$M_0(\tau)=-\frac{\sum_{n=0}^{\infty}\left[A_n\Psi(\mu_n)\Big/J_1(\mu_n k)\right]e^{-\mu_n^2\tau}}{\sum_{n=0}^{\infty}\left[A_n\Psi(\mu_n)\Big/\mu_n J_1(\mu_n k)\right]e^{-\mu_n^2\tau}} \quad (25)$$

As $\tau\to\infty$, asymptotic representations for f_0 and M_0 are given by

$$f_0(\infty,r)=\frac{\left(1-k^2\right)}{2}\Psi(\mu_0) \text{ and } M_0(\infty)=-\mu_0^2 \quad (26)$$

Using (11) evaluating M_1 we get

$$M_1=\frac{-4\mu_0 E_0(\mu_0)\int_k^1 rw(r)E_0(\mu_0\gamma)f_0 dr}{\left(1-k^2\right)\left(\mu_0^2+\beta^2\right)\{E_0(\mu_0)\}^2--k^2\mu_0^2\{E_0(\mu_0 k)\}^2} \quad (27)$$

Again, applying a similar argument as used to get M_1 the dispersion coefficient is obtained as

$$M_2-\frac{1}{Pe^2}=\frac{-\int_k^1 r\left[w(r)+M_1\right]f_1E_0(\mu_0 r)dr}{\int_k^1 rE_0(\mu_0 r)f_0 dr} \quad (28)$$

Results And Discussions : In the present analysis, the development of the dispersive transport following the insertion of a chemically active tracer in a solvent flowing through an annulus with reactive outer wall has been studied. The effect of moderate wall absorption, in the presence of coaxial inner tube and the effect of couple stress on the three effective transport coefficient, viz, the exchange coefficient, the convection and the dispersion coefficients are examined.

The value of the wall absorption parameter β is taken to range from 0.01 to 100, and the ratio of the inner tube to the outer tube, k is varied from 0.1 to 0.4. For each value of β and k, the associated eigen values μ_n, are obtained by a standard root finding procedure. The objective is to evaluate the exchange coefficient $M_0(\tau)$, the asymptotic coefficient M_1 and the asymptotic dispersion coefficient M_2, and to find their dependence on the size of the annular gap k and the wall absorption parameter β. In fact, most of the graphical representations show the variations of these transfer coefficients with changes in the values of β, the absorption parameter.

Unlike the Taylor's dispersion where there is a conservation of mass of solute in the gaseous phase, the total amount of solute is no longer a conserved quantity because of the irreversible reaction which occurs at the tube wall. The coefficient $M_0(\tau)$, accounting for this non zero solute flux at the tube wall, will be negative in this problem to account for the depletion of solute in the system. The root μ_0 is of particular importance since the total amount of contaminant cloud decays as $e^{-\mu_0\tau}$, and also large times the absorption coefficient $M_0=\mu_0{}^2$.

Figure (2) describes the variation of $-M_0$ (the absorption coefficient) with β and k for large times. $-M_0$ increases from 0.0202 to 5.91 as the absorption parameter β increases from 0.01 to 100 for k as small as 0.1. as β increases, the reaction at the wall consumes the material more rapidly than it can be supplied by molecular diffusion. Thus the mass transport system becomes diffusion controlled. Also, it can be seen form figure (2), that $-M_0$ increases from 5.78 in the case of a tubular flow (k=0) to 9.24 for flow in an annulus (k=0.4) for large β ($\beta=100$). Thus, there is more absorption of solutes at the wall in annulus compared to the tubular flow.

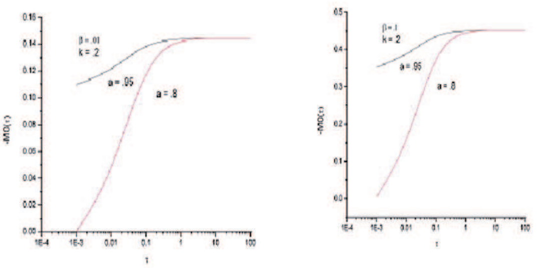

Figure 2. Plot of exchange coefficient vs time for different values of absorption coefficient

Figure (3) and (4) shows the asymptotic convection coefficient, $-M_1$, in an annulus against the wall absorption parameter β for different k. $-M_1$

decreases from 0.379 (k=0.2) to 0.1951 (k=0.5) with the decrease of the annular gap for $\beta = 100$. - M_1 also decreases with increase in couple-stress parameter α. showing that, mass transfer is accelerated with inclusion of spin of suspension.

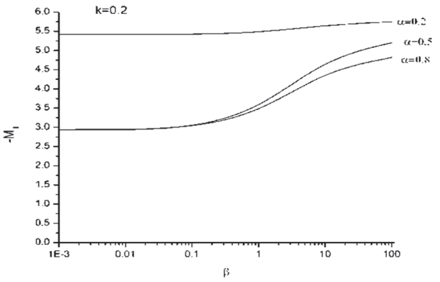

Figure 3 Plot of Convective coefficient vs absorption coefficient

Figure (5) shows the asymptotic dispersion coefficient, M_2 (from which the additive contribution of the axial diffusion$\frac{1}{Pe^2}$ has been deducted) against the wall absorption parameterβ, for different k. it is also seen that M_2 decreases with an increase in the wall absorption parameterβ. The dispersion coefficient also shows the same behavior as convection coefficient.

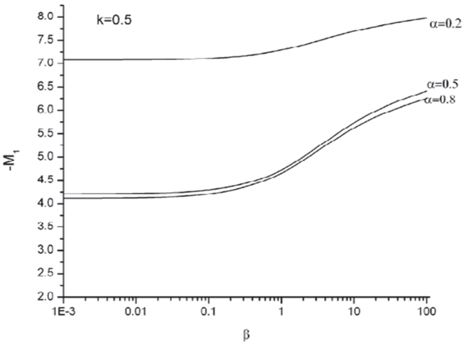

Figure 4 Plot of convective coefficient vs absorption coefficient

It decreases with increase in couple stress parameter α. Couple stress parameter is inversely proportional to spin of molecules indicating mass transfer is enhanced by consideration of couple stress.

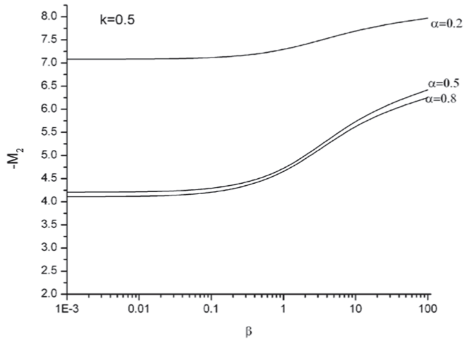

Figure 5 Plot of dispersion coefficient vs absorption

Conclusions : The dispersion of a solute in an annular region is studied using the generalized dispersion model. The irreversible absorption characteristic of the wall is considered. The study brings out the development of the dispersive transport following the injection of a tracer in terms of the three effective transport coefficients, viz. the exchange coefficient, the convention and the dispersion coefficients. It is observed that the asymptotic absorption coefficient, - M_0, increases from 5.78 in the case of a tubular flow (k=0) to 9.24 for the flow in the annulus (k=0.4) for large β ($\beta = 100$). Thus, the absorption of solutes at the wall is favored in an annulus. The asymptotic dispersion coefficient decreases with an increase in the wall absorption parameterβ. It decreases form 1.49×10^{-3} ($\beta = 0.01$) to 8×10^{-4} ($\beta = 100$) for k=0.1. The inclusion of couple stress increases convection as well as dispersion and does not affect absorption by wall. The effect of pressure of couple stress is to increase mass transfer.

Acknowledgment The Authors thank VTU research Grant Scheme and Management of NMIT and management of Saptagiri Institute of Technology, Bangalore for their support in carrying this work.

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Indira R Rao, Department of Mathematics, NMIT, Bangalore, India/Professor/
indira_rao09@rediffmail.com .,
Jayaprakash M C,Department of Mathematics, Saptagiri Institute of Technology , Bangalore , India/
Jayaprakash_blr@yahoo.co.in