ADOMIAN APPROXIMATION SOLUTION FOR FRACTIONAL ORDER RICCATI DIFFERENTIAL EQUATION

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Abstract : In this article, the Adomian Decomposition method (ADM) is applied to obtain the solution of fractional η order (where $\eta(>0) \in \mathbb{R}$) Riccati differential equation $D_t^{\eta} y(t) = P(t) + Q(t)y(t) + R(t)y(t)^2$. The equation under consideration has been examined using one with variable coefficient and another with constant coefficient. The numerical results obtained by this method have been shown graphically which confirms the plausible feature that Adomian Decomposition method is a powerful technique for the solution of nonlinear fractional Riccati differential equation. Most of the symbolic and numerical computations have been performed using Wolfram Mathematica 7 software.

Keywords : Adomian Decomposition Method, Fractional Riccati Differential Equation, Riemann-Liouville fractional derivative, Caputo derivative.

Introduction : The Adomian Decomposition method (ADM) is a semi-analytical method for solving ordinary and partial nonlinear differential equations. The method was developed from the 1970's to the 1990's by George Adomian. The vital aspect of the method is employment of the "Adomian Polynomials" which allow for solution convergence of the nonlinear portion of the equation, without simply linearizing the system. This algorithm provides the solution in a rapidly convergent series. The ADM has successfully been applied to many situations like S. Saha Ray et al [5]-[8] used ADM to obtain approximate solutions for extraordinary differential.

The Riccati equation (RE), named after the Italian mathematician Jacopo Francesco Riccati, is a basic first-order nonlinear ordinary differential equation (ODE) that arises in different fields of mathematics and physics. It has the general form

$$D_t y(t) = P(t) + Q(t)y(t) + R(t)y(t)^2 \qquad (1.1)$$

It is assumed that $y(t)$, $P(t)$, $Q(t)$ and $R(t)$ are analytic functions of the real argument t.

Mathematical Preliminaries Of Fractional Calculus :

The notion of fractional calculus was first anticipated by Leibnitz, one of the founders of standard calculus, in a letter written in 1695 [4]. This calculus involves different definitions of the fractional operators such as Riemann–Liouvillefractional derivative, Caputo derivative, and Grunwald–Letnikov fractional derivative [4]-[5]. The fractional calculus has gained considerable importance during the past decades mainly due to its applications in diverse fields of science and engineering. For the purpose of this paper the Caputo's definition of fractional derivative will be used.

Definition-Caputo Fractional Derivative

The fractional derivative introduced by Caputo in the late sixties, is called Caputo Fractional Derivative. The fractional derivative of f(t)in the Caputo sense is defined by

$$D_t^{\alpha} f(t) =$$

$$J^{m-\alpha} D^m f(t) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t (t-\tau)^{(m-\alpha-1)} \dfrac{d^m f(\tau)}{d\tau^m} d\tau, & if\, m-1<\alpha<m,\ m\in\mathbb{N} \\ \dfrac{d^m f(t)}{dt^m}, & if\, \alpha=m, m\in\mathbb{N} \end{cases}$$

where the parameter α is the order of the derivative and is allowed to be real or even complex. In this paper only real and positive α will be considered.

For the Caputo's derivative we have

$D^{\alpha} C = 0$, (C is a constant)

$$D^{\alpha} t^{\beta} = \begin{cases} 0, & \beta \le \alpha - 1 \\ \dfrac{\Gamma(\beta+1) t^{\beta-\alpha}}{\Gamma(\beta-\alpha+1)}, & \beta > \alpha - 1 \end{cases}$$

Similar to integer order differentiation Caputo's derivative is linear.

$$D^{\alpha}(\gamma f(t) + \delta g(t)) = \gamma D^{\alpha} f(t) + \delta D^{\alpha} g(t)$$

where γ and δ are constants, and satisfies so called Leibnitz's rule.

$$D^{\alpha}(g(t) f(t)) = \sum_{k=0}^{\infty} \binom{\alpha}{k} g^{(k)}(t) D^{\alpha-k} f(t)$$

If $f(\tau)$ is continuous in [0,t] and $g(\tau)$ has continuous derivatives sufficient number of times in [0, t].

Adomian Decomposition Method (Adm) :

Let us consider the general form of a differential equation [1]

$$F\ y = g \qquad (3.1)$$

whereF is the non-linear differential operator with linear and non-linear terms. The differential operator is decomposed as

$$F \equiv L + R \qquad (3.2)$$

where L is easily invertible linear operator and R is the remainder of the linear operator. For our convenience L is taken as the highest order derivative then the eq. (3.1) can be written as

$$Ly + Ry + Ny = g \quad (3.3)$$

whereNy corresponds to the non-linear term. Solving Ly from (3.3), we have

$$Ly = g - Ry - Ny \quad (3.4)$$

Because L is invertible, then L^{-1} is to be the integral operator

$$L^{-1}(Ly) = L^{-1}(g) - L^{-1}(Ry) - L^{-1}(Ny) \quad (3.5)$$

If L is a second order operator, then L^{-1} is a two-fold integral operator

$$L^{-1} \equiv \int_0^t \int_0^t (\bullet) dt\, dt \text{ and } L^{-1}(Ly) = y(t) - y(0) - ty'(0)$$

Then eq. (3.5) for y yields,

$$y = y(0) + ty'(0) + L^{-1}(g) - L^{-1}(Ry) - L^{-1}(Ny) \quad (3.6)$$

Let us consider the unknown function $y(t)$ in the infinite series as

$$y(t) = \sum_{n=0}^{\infty} y_n . \quad (3.7)$$

The non-linear term N(y) will be decomposed by the infinite series of Adomian polynomials $A_n\ (n \geq 0)$ [1]

$$Ny = \sum_{n=0}^{\infty} A_n \quad (3.8)$$

where A_n's are obtained by

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} N\left(\sum_{i=0}^{\infty} y_i \lambda^i\right)\right]_{\lambda=0} \quad (3.9)$$

Now, substituting (3.7) and (3.8) into (3.6), we obtained

$$\sum_{n=0}^{\infty} y_n = y(0) + ty'(0) + L^{-1}(g) - L^{-1}\left[R\left(\sum_{n=0}^{\infty} y_n\right)\right]$$

$$- L^{-1}\left[\left(\sum_{n=0}^{\infty} A_n\right)\right] \quad (3.10)$$

Consequently we can obtain,

$$y_0 = y(0) + ty'(0) + L^{-1}(g)$$

$$y_1 = -L^{-1}R(y_0) - L^{-1}(A_0)$$

....

$$y_{n+1} = -L^{-1}R(y_n) - L^{-1}(A_n) \quad (3.11)$$

and so on.

Based on the Adomian Decomposition method, we shall consider the solution $y(t)$ as

$$y \cong \sum_{k=0}^{n-1} y_k = \varphi_n \text{ with } \lim_{n\to\infty} \varphi_n = y(t)$$

In most of the problems, the practical solution φ_n the n-term approximation is convergent and accurate even for small value of n.

Analysis For Fractional Riccati Differential Equation

In this present analysis, we present analytical approximate solutions for the fractional order Riccati differential equations given by

$$D_t^{\eta} y(t) = P(t) + Q(t)y(t) + R(t)y(t)^2 \quad (4.1)$$

D_t^{η} is the Caputo fractional derivative of order η ($\eta(>0) \in$ ✡)).

HereQ(t), R(t), P(t) and G(t) are analytical functions. To solve (4.1) by means of Adomian Decomposition method, we require Adomian polynomials A_n given by the formula eq. (3.9).

Applying the fractional integral operator J $^{\eta}$ both sides of eq. (4.1), we have

$$y(t) = y(0) + ty'(0) + J^{\eta} P(t) + J^{\eta} (Q(t)y(t)) + J^{\eta} \left(\sum_{n=0}^{\infty} A_n\right) \quad (4.2)$$

Numerical Examples And Discussions

Consider the following example which is the fractional η order nonlinear Riccati differential equation with coefficients [2]

$$D_t^{\eta} y + 2\beta y^2 - \alpha = 0, \text{ with } 0 < \eta < 1 \quad (5.1)$$

Case-1: Here we consider, the constant coefficients with initial condition $y(0) = 1$

Also we consider, the constant coefficient as function of arbitrary parameter given by $\beta = \frac{1}{2\cos\delta}\left(e^{-\delta+1}\right)^3$

and $\alpha = \left(\frac{-(1+e^{-\delta})\sin\delta}{\left(1+\left(\frac{\sqrt{2}(1-e^{1\delta}+\delta)}{\delta}\right)\right)^2}\right)$ where δ is some arbitrary parameter.

Now, the Adomian polynomials can be obtained as

$$A_0 = y_0^2,\ A_1 = 2y_0 y,\ A_2 = y_1^2 + 2y_0 y,$$

$$A_3 = 2y_0 y_3 + 2y_1 y_2$$

In the similar manner, we can construct rest of all the Adomian polynomials. By using eq. (4.2), we have

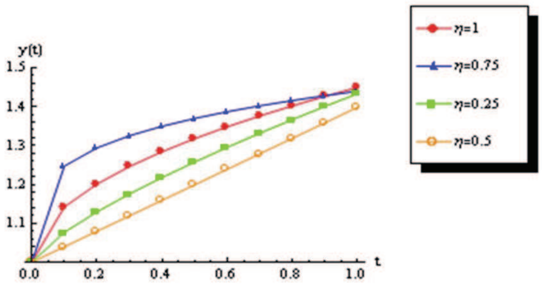
$$y_0 = 1,\ y_1 = \frac{t^{\eta}}{\Gamma(1+\eta)}(-2\beta + \alpha),$$

$$y_2 = \left(\frac{-4\beta(-2\beta+\alpha)}{\Gamma(2\eta+1)}\right)t^{2\eta} + \alpha\left(\frac{t^{\eta}}{\Gamma(\eta+1)}\right) \text{ and}$$

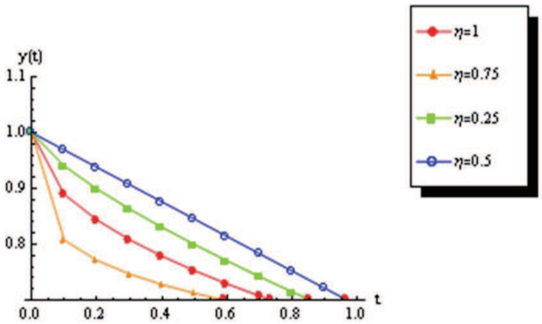
$$y_3 = -2\beta\left(\frac{(-2\beta+\alpha)}{\Gamma(1+\eta)}\right)^2 \frac{t^{3\eta}\Gamma(2\eta+1)}{\Gamma(3\eta+1)} + \left(\frac{16\beta^2(-2\beta+\alpha)}{\Gamma(3\eta+1)}\right)t^3 - \left(\frac{4\beta\alpha}{\Gamma(2\eta+1)}\right)t^{2\eta} + \alpha\left(\frac{t^\eta}{\Gamma(\eta+1)}\right)$$

and so on. Hence, the approximate solution $y(t)$ is $y(t) = y_0 + y_1 + y_2 + y_3 + \dots$.

By considering, the fractional order with different parameter values δ , the approximate solutions are cited byFigers1-2.



Figer 1 Approximation solutions with arbitrary parameter $\delta = 5$



Figer2 Approximation solutions with arbitrary parameter $\delta = 15$

Case-2: Let us consider the variable coefficient with initial condition $y(0) = 0$ for eq.

(5.1)

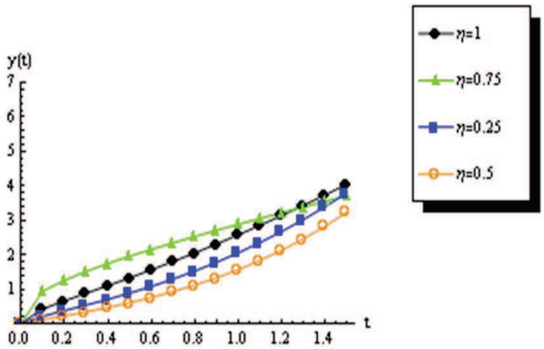
Here we consider, the variable coefficient as function of time given by $\beta = -t\cos\delta$ and $\alpha = \left(\frac{t^{\delta^2}\delta^2\cos\delta}{\delta^2 + 4\sqrt{2}\delta\sin\delta + 8\sin^2\delta}\right)$. Here, the Adomian polynomials are same as in Case-1 and by using eq. (4.2) we have $y_0 = 0$,

$$y_1 = \frac{t^{\eta+1}}{\Gamma(2+\eta)}(2\cos\delta) + \frac{t^{\eta+\delta^2}\delta^2\cos\delta}{\delta^2 + 4\sqrt{2}\delta\sin\delta + 8\sin^2\delta}\left(\frac{\Gamma(\delta^2+1)}{\Gamma(\delta^2+\eta+1)}\right)$$

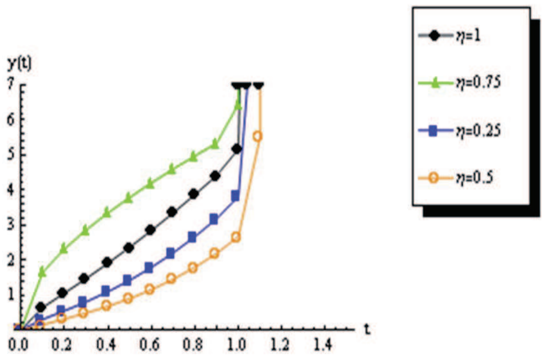
$$y_2 = \frac{t^{2\eta+\delta^2}\delta^2\cos\delta}{\delta^2 + 4\sqrt{2}\delta\sin\delta + 8\sin^2\delta}\left(\frac{\Gamma(\delta^2+1)}{\Gamma(\delta^2+2\eta+1)}\right)$$

and so on. Hence, the approximate solution $y(t)$ is $y(t) = y_0 + y_1 + y_2 + \dots$.

In case of variable coefficient approximate solutions are cited by Figers3-4 for different parameter values δ



Figer3 Approximation solutions with arbitrary parameter $\delta = 1$.



Figer4 Approximation solutions with arbitrary parameter $\delta = 7$.

Conclusion : In the present paper, Adomian Decomposition method (ADM) has been successfully applied for solving fractional Riccati differential equation with constant and variable coefficients. This efficient technique applied to linear or nonlinear problem is particularly valuable tool for Scientists and Applied Mathematicians. The computations associated with the exhibited examples in this paper have been performed using Mathematica software.

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