ON ALGEBRAIC STRUCTURES

N. MOTAHARI, T. ROUDBARI

Abstract : In this paper, by considering the notion of some algebraic structures, we have obtained some relations between abelian groups as Z- modules, modules, BCK- modules and MV- modules.

Keywords: BCK-algebra, BCK-module, MV- algebra, MV- module, module

Introduction : In 1966, Imai and Iseki [7] introduced BCK-algebras. This notion was originated from two different ways: (1) set theory, and (2) classical and no classical propositional calculi. Today BCK-algebras have been applied to many branches of mathematics, including group theory, functional analysis, probability theory, topology, fuzzy set theory. Every module is an action of ring on certain group. This is, indeed, a source of motivation to study the action of certain algebraic structures on groups. BCK-module is an action of BCK-algebra on commutative group. In 1994, the notion of BCK-module was introduced by M. Aslam, H. A. S. Abujabal and A. B. Thaheem [2]. They established isomorphism theorems and studied some properties of BCK-modules. The theory of BCK-modules was further developed by Z. Perveen and M. Aslam [9]. Also in 2003, the notion of MV-module was introduced by A. Di Nola, P. Flondor and I. Leustean as an action of MV-algebra on PMV-algebra [5].Now, we have obtained some relations between abelian groups as Z- modules, modules, BCK-modules and MV- modules.

BCK-Algebras: According to Imai and Iseki [7], a BCK-algebra is a non-empty set X with a binary operation *
and with a constant element 0 satisfying the following axioms: for all x, y, z ∈ X

(BCK1)((x * y)* (x * z)) * (z * y) = 0,

(BCK2) (x * (x * y)) * y = 0,

(BCK3) x * x = 0,

(BCK4) 0 * x = 0,

(BCK5) x * y = y * x = 0 imply that x = y, for all x, y, z ∈ X.

If there is an element 1 of a BCK- algebra X, satisfying x * 1 = 0, for all x in X, the element1 is called unit of X. A BCK- algebra with unit is called to be bounded.

In a BCK-algebra (X, *, 0) we can define a partial binary operation ≤ by x ≤ y if and only if $x * y = 0$.

For all x ∈ X we have x * 0 = x.

For any x, y ∈ X denote $x \wedge y := y * (y * x)$.

Obviously, $x \wedge y$ is a lower bound of x and y. Also $x \wedge x = x, x \wedge 0 = 0 \wedge x = 0$.

Example2.1. [8] Let X be a partially ordered set with the least element 0. The operation * on X is defined by $x * y = \begin{cases} 0 & if\, x \le y \\ 0 & otherwise \end{cases}$

Then (X, *, 0) is a BCK-algebra. Hence any partially ordered set is regarded as a BCK- algebra.

BCK-Modules: According to Abujabal, Aslam and Thaheem [1], a BCK-module is an action of a BCK-algebra
on abelian group (M, +).

Let (X, *, 0) be a BCK-algebra. Then a left X-module M over the BCK-algebra X, consists of an abelian group (M, +) and an operation (.): X×M→M such that

(i) (x ∧ y) · m = x · (y · m),

(ii) x · $(m_1 + m_2)$ = x · m_1+ x · m_2,

(iii) 0 · m = 0,

for all x, y ∈ X, m_1, m_2∈ M, where x ∧ y = y * (y * x).

If X is bounded, then the following additional condition holds:

(iv)1 · m = m.

A right X-module can be defined similarly. For the rest of the paper, all the BCK- modules will be left BCK - modules.

Example3.1. [1] Let A be a non-empty set and X = P (A) be the power set of A. Then X is a bounded commutative BCK-algebra with x ∧ y = x - y, for all x, y ∈ X. Define x + y = (x ∪ y)∩ $(x \cap y)'$, the symmetric difference. Then M = (X, +) is an abelian group with empty set ∅ as an identity element and x + x = ∅. Define x · m = x ∩ m, for any x, m∈ X. Then simple calculations show that:

(i) (x ∧ y) · m = (x ∩ y) ∩ m = x ∩ (y ∩ m) = x · (y · m),

(ii) x · $(m_1+ m_2)$ = x · m_1+ x · m_2,

(iii) 0 · m = ∅ ∩ m = ∅ = 0,

(iv) 1 · m = A ∩ m = m. Thus X itself is an X-module.

MV- Algebras And PMV-Algebras : An MV-algebra is a structure (M, ⊕, ∗, 0), where ⊕ is a binary operation, ∗ is a unary
operation, and 0 is a constant such that the following axioms are satisfied for any a, b ∈M:

(MV1) (M,⊕, 0) is an abelian monoid,

(MV2) $(a^*)^*$= a,

(MV3) 0^*⊕a =0^*,

(MV4) $(a^* \oplus b)^*$⊕ b = $(b^* \oplus a)^*$ ⊕ a [5].

Definition4.1. [5] A product MV-algebra (or PMV-algebra, for short) is a structure
(A, ⊕, ∗, ·, 0), where (A,⊕, ∗, 0) is an MV-algebra and · is a binary associative operation on A such that the

following property is satisfied:
if x + y is defined, then x · z + y · z and z · x + z · y are defined and
(x + y) · z = x · z + y · z, z · (x + y) = z · x + z · y, where + is the partial addition on A.
Example4.2. [5] {0,1} is a PMV-algebra.
MV-Modules : By [5], let (A, ⊕, ∗, ·, 0) be a PMV-algebra and (M,⊕, ∗, 0) an MV-algebra. We say that M is a (left) MV-module over A (or, simply, A-module) if there is an external operation ϕ :A ×M →M, ϕ(α, x) = αx, such that the following properties hold for any x, y ∈M and α,β ∈ A:
(1) if x + y is defined in M then αx + αy is defined and α(x + y) = αx + αy,
(2) if α + β is defined in A then αx + βx is defined in M and
(α + β) x = αx + βx,
(3) (α · β) x = α (βx).
We say that M is a unital MV-module if A is a unital PMV-algebra and M is an MV-module over A such that
(4) 1_A x = x for any x ∈M.
We remark that we can define the notion of right MV-module if we consider the external
Operation ϕ: M ×A→M, ϕ (x,α) = xα, which satisfies the right version of the axioms from Definition, for the rest of the paper, all the MV-modules will be left-MV-modules.
Example5.1. [5] Every MV- algebra is a {0, 1}-module.
Modules : Every module is an action of a ring on abelian group (M, +) [6].
A left R-module M over the ring R consists of an abelian group (M, +) and an operation
(.) : R ×M → M such that for all r, s in R and x, y in M, we have:
(i) r. (x + y) = r. x + r. y
(ii) (r + s). x = r. x + s. x
(iii) (rs).x = r.(s.x)
1_R . x = x if R has multiplicative identity1_R. A right R-module can be defined similarly.
Example6.1. [6] Every abelian group is a Z-module.

Main Results:

BCK- modules and Z-modules
Lemma: Every Z-module (M) has a BCK- module structure.
Proof: Let N = {0, 1, 2, ...} and A={a_n: n∈N}. Denote X=N∪A in which the operation * is defined as follows:
For n, m ∈ N,

$$n * m = \begin{cases} 0, & if\ n < m \\ n\text{-}m & if\ n \geq m \end{cases},$$

n * a_m = n * b_m = 0,
b_m* n = b_{n+m},
a_m* n = a_{m+n},

$$a_n * a_m = b_n * b_m = \begin{cases} 0, & if\ n < m \\ n\text{-}m & if\ n \geq m \end{cases},$$

a_n* b_m = a_n* a_{m+1} ,
b_n*a_m = b_n * b_{m+1}.
Define a ≤ $b\ if$ and only if a * b = 0. It is easily seen that $(N \cup A, \leq)$ is a partially ordered set. Also it is easy to see that (N ∪ A, *, 0) is a BCK- algebra [8]. We consider N = {0, 1, 2, ...} and
A = {0, -1, -2, ... }. Then we have $N \cup A = Z$. So $(Z,*,0)$ is a BCK - algebra.
Now we define the map (.): Z× M → M by

$$x * n = \begin{cases} n & if\ n < m \\ 0 & if\ n \geq m \end{cases},$$

It is clear that (.) is well defined . We show that (x ∧ y).m = x. (y. m), for all $x, y \in Z$ and for all $m \in M$. we consider the following cases :

1- Let $x \in N$ and $y \in N$. Then $\wedge\ y = \begin{cases} y & if\ y < x \\ x & if\ y \geq x \end{cases}$.

2- Let $x, y \in A$. Then there exist $m, n \in N$ such that $x = a_n$ and $y = a_m$.

$$a_m \wedge a_n = \begin{cases} a_n & if\ m < n \\ a_m & if\ m \geq n \end{cases}$$

3- Let $x \in N$ and $y \in A$. Then there exists $m \in N$ such that $y = a_m$. Also $n \wedge a_m = a_m \wedge n = n$
It is easy to see that in the all of cases $(x \wedge y).m = x.(y.m)$, for all $m \in M$.
The other properties are obvious, hence M is a Z-module (BCK- module).
BCK- modules and modules
Lemma: Every module (M) on a ring (R) has a BCK-module structure.
Proof: Let R be an arbitrary ring. Then we have (P (R), - , ∅) is a BCK-algebra.
Now we define the map (.): P (R)× M → M by

$$A \cdot n = \begin{cases} n & if\ x_0 \in A \\ x & if\ x_0 \notin A \end{cases},$$

where x_0 is an element of R.
It is clear that (.) is well defined. At first we show that (A ∧ B). m = A. (B. m),
for all A,B ∈ X and for all m ∈ M. we consider the following two cases and their related sub cases:
Case 1: Let x_0 ∈ A ∩ B. Then we get x_0 ∈ A and x_0 ∈ B. So (A ∧ B). m = A. (B. m) = m.
Case 2: Let x_0 ∉ A ∩ B. Then we have 3 sub cases:
Sub case 1: x_0 ∈ A and x_0 ∈ B.
Sub case 2: x_0 ∉ A and x_0 ∉ B.
Sub case 3: x_0 ∈ A and x_0 ∉ B.
In all of this cases
(A ∧ B). m = A. (B. m) = 0
It is clear that
A . $(m_1 + m_2)$ = A.m_1 + A.m_2,

for all A∈ X,m_1,m_2 ∈ M. And 0.m = 0. Hence M is a P(R) - module.

BCK- modules and MV- modules

Lemma: Let M be a BCK – module. Then P (M) has a MV- module structure.

Proof: Let M be a BCK- module. Then (P (M), -, ∅) is a bounded and commutative BCK- algebra, that has a MV-algebra structure.

Now by Examples 4.2 and 5.1, P (M) has a MV-module structure ({0, 1}- module).

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Dept. of Math., Islamic Azad University, Kahnooj Branch, Kerman, Iran
E – mail: narges.motahari@yahoo.com
Dept. of Math., Islamic Azad University, Kerman Branch, Kerman, Iran
E – mail: Taherehroodbarylor@yahoo.com