ON A DISCRETE VERSION OF BILATERAL GAMMA DENSITY

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Abstract : We introduce a new distribution called discrete bilateral gamma distribution as a discrete version of bilateral gamma distribution. Discrete Laplace distribution can be considered as a particular case of this distribution. In this paper we consider some basic distributional as well as moment properties. Estimation of the parameters of the distribution is also addressed. We conduct an empirical study of the proposed distribution using simulated data.

Keywords: Bilateral gamma density, Discrete bilateral gamma distribution, Discrete Laplace distribution.

Introduction : Usual count data models, such as Poisson distribution and negative binomial distribution can only cover zero and positive integer values. Discrete distributions which are defined over values of positive and negative integers are rare in literature. Recently there has been much research to construct discrete versions of continuous distributions. Discrete normal distribution developed by Lisman and Zuylen (1972) and studied in detail by Kemp (1997) can be used to study count data supported on the set of integers {0, ±1, ±2, ...} . Following Kemp (1997) discrete versions of Laplace distribution and their skewed generalizations were developed by Inusah and Kozubowski (2006).

Kuchler and Tappe (2008a,b) developed bilateral gamma distribution as the distribution of the differences of independently distributed gamma random variables. This distribution is a mixture of normal distribution with stochastic variance having gamma distribution. The tails of the bilateral gamma distribution decrease more slowly than the normal distribution. They are infinite divisible, self decomposible and are stable under convolution. Due to these properties this model is recently much popular in financial modeling. Bilateral gamma distribution have application in different contexts such as growth of melatonin in human body, growth-decay mechanisms like formation of sand dunes , formation of solar neutrino fluxes in cosmos, input-output process in econometric contexts and industrial productions etc.

A bilateral gamma distribution with parameters $\alpha^+, \lambda^+, \alpha^-, \lambda^-$ >0 is defined as the distribution of X= X_1 – X_2 , where X_1 and X_2 are independent gamma random variables with parameters (α^+, λ^+) and (α^-, λ^-) respectively.The probability density function (p.d.f) can be obtained as the convolution of two gamma densities for $x\epsilon\ (0,\infty)$,as

$$f(x) = \frac{(\lambda^+)^{\alpha^+}(\lambda^-)^{\alpha^-}}{(\lambda^+ + \lambda^-)^{\alpha^-}\Gamma(\alpha^+)\Gamma(\alpha^-)} e^{-\lambda^+ x} \int_0^\infty v^{\alpha^- -1}(x + \frac{v}{\lambda^+ + \lambda^-})^{\alpha^+ -1}\, e^{-v}\, dv \qquad (1.1)$$

This density can be expressed in terms of Whittaker function $W_{\lambda,\mu}(z)$ as follows

$$f(x) = \frac{\llbracket(\lambda\rrbracket^+)^{\alpha^+}\llbracket(\lambda\rrbracket^-)^{\alpha^-}}{(\llbracket\lambda^+ + \lambda^-)\rrbracket^{\llbracket\frac{1}{2(\alpha\rrbracket^+}+\alpha^-)}\Gamma(\alpha^+)} x^{\llbracket\frac{1}{2(\alpha\rrbracket^+}+\alpha^-)-1} e^{\llbracket\frac{-x}{2(\lambda\rrbracket^{\pm\lambda}}^-)}$$

$$W_{\llbracket\frac{1}{2(\alpha\rrbracket^{\pm\alpha}}^-),\ \llbracket\frac{1}{2(\alpha\rrbracket^+}+\alpha^- -1),}\left(x\llbracket(\lambda\rrbracket^+ + \lambda^-)\right)$$

where the Whittaker function has the representation

$$W_{\lambda,\mu}(z) = \frac{z^\lambda e^{-\frac{z}{2}}}{\Gamma(\mu-\lambda+\frac{1}{2})} \int_0^\infty t^{\mu-\lambda-\frac{1}{2}}\, e^{-t}\, (1+\frac{t}{z})^{\mu+\lambda-\frac{1}{2}}\, dt$$

for $\mu - \lambda > \frac{1}{2}$

Sato et al. (1999) proposed discrete exponential distribution as geometric distribution with p.d.f.

$$P(Y = k) = (1-q)q^k, \qquad k = 0,1,2\ldots$$

The convolution of independent exponential random variables follows the gamma distribution. Then the convolution of discrete exponential is termed as discrete gamma distribution (Sato 1999),but which is same as the negative binomial distribution with p.d.f.

$$P(Y = k) = \binom{r+k-1}{k} q^r k \quad , k = 0,1,2\ldots \quad \text{where} \quad q = e^{-\lambda} \qquad (1.2)$$

where q is the probability of success of an event and k is the number of failures before the r^{th} success such that p = (1-q).

Hence the distribution of the difference of two negative binomial random variables can be considered as the distribution which corresponds to the discrete version of the difference of two gamma random variables. This leads to the discrete version of bilateral gamma distribution.

The rest of the paper is organized as follows, the derivation of the probability density function of discrete bilateral gamma distribution denoted by DBG(r, p) is given in section 2 and distributional properties were discussed in section 3, the estimation process and an empirical study is given in section 4.

Discrete Bilateral Gamma Density

Let X be a negative binomial random variable with p.d.f. given by (1.2).Then the characteristic function of X is given by,

$$\varphi_X(t) = \frac{(1-p)^r}{(1-pe^{it})^r} \quad where \quad |pe^{it}| < 1 and \ t \in \mathbb{R} \qquad (2.1)$$

Let X_1 and X_2 are two independent negative binomial random variables with same parameters r and q. Let, $Y = X_1 - X_2$. Then the characteristic function of Y is obtained as

$$\varphi_Y(t) = \left[\frac{(1-p)^2}{(1-pe^{it})(1-pe^{-it})}\right]^r \qquad (2.2)$$

The probability generating function of Y is given by

$$P(s) = \left[\frac{(1-p)^2}{(1-ps)\left(1-\frac{p}{s}\right)}\right]^r \qquad (2.3)$$

Then the distribution of Y corresponding to (2.2) is referred as discrete bilateral gamma distribution with parameters r and p , denoted by DBG(r,p) and its probability density function is given by

$$f(y) = P(Y = m) = (1-p)^{2r} \sum_{k=|m|}^{\infty} \binom{r+k-1}{k} \binom{r+k-|m|-1}{k-|m|} p^{2k-|m|} \qquad (2.4)$$

$m = 0, \pm 1, \pm 2, 3, \ldots \ where\ p = e^{-1/\sigma},\ r > 0$

In terms of Hyper Geometric function, the p.m.f. can be expressed as

$$f(y) = P(Y = m) = \frac{(1-p)^{2r} p^{|m|}\ \Gamma(r+|m|)\ {}_2F_1\{(r, r+|m|), (1+|m|), p^2\}}{\Gamma r\ \ \Gamma(1+|m|)} \qquad (2.5)$$

When r=1,

$$P(Y = m) = \frac{1-p}{1+p} p^{|m|}, m = 0, \pm 1, \pm 2, \ldots.$$

which is the p.d.f of the symmetric discrete Laplace distribution,DL(p) introduced by Kozubowski and Inusah (2006). Hence discrete Laplace distribution can be considered as a particular case of discrete bilateral gamma distribution given by (2.4).

The following figures are the plots of the p. d. f. corresponding to different values of r and p which will give an idea about the shape and peakedness of the distribution. Since the DBG(r,p) distribution is symmetric, it is unimodal and mode is zero.

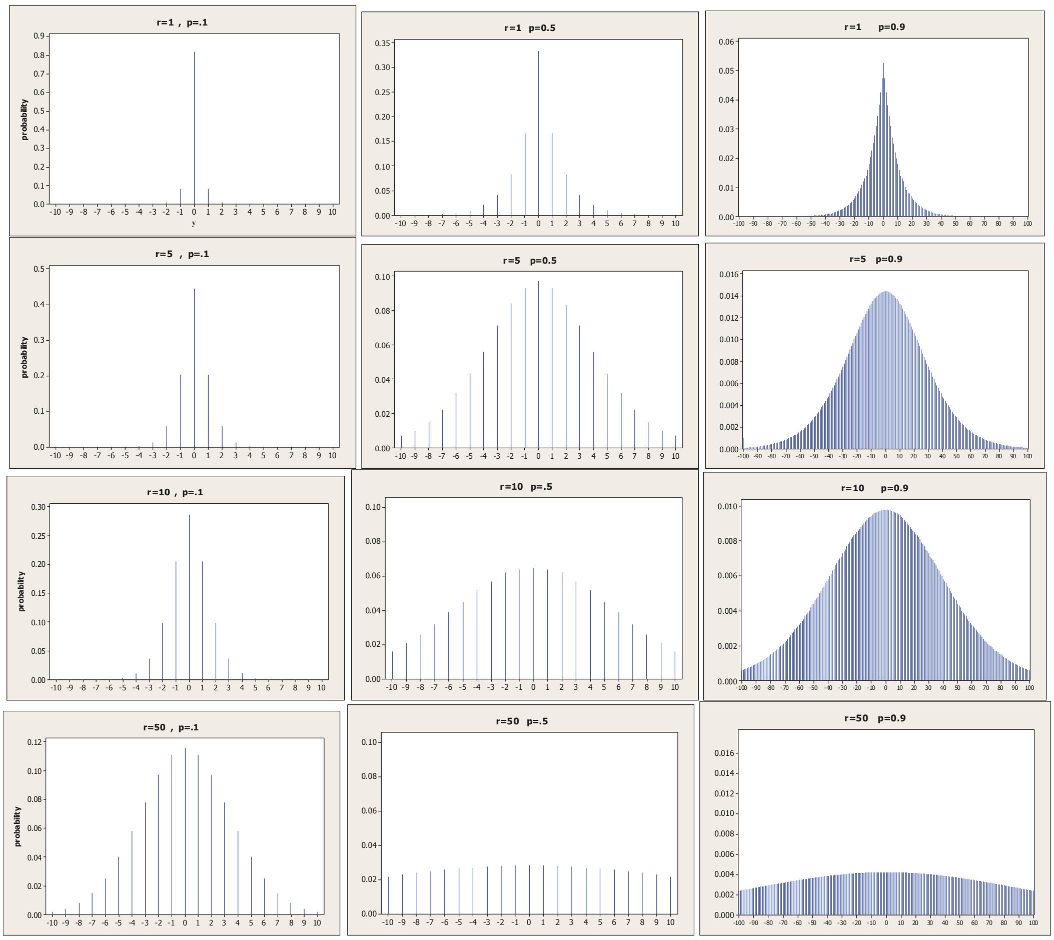

Fig. 1:Discrete bilateral gamma density graph for different values of r and p.

Distribution Properties

Distribution Function

Let Y follows DBG (r,p) and m ≤ y < m+1, where m is an integer Then

$$F(y) = P(Y \le y) = \sum_{j=-\infty}^{m} P(Y = j)$$

Case I When y < 0 , then m is negative integer

$$F(y) = \sum_{j=-\infty}^{m} (1-p)^{2r} \sum_{k=|j|}^{\infty} \binom{r+k-1}{k} \binom{r+k-|j|-1}{k-|j|} p^{2k-|j|}$$

$$F(y) = (1-p)^{2r} \sum_{j=-m}^{\infty} \sum_{k=j}^{\infty} \binom{r+k-1}{k} \binom{r+k-j-1}{k-j} p^{2k-j} \qquad (3.1)$$

Case II When y ≥0 then m+1 is a positive integer.

F(y) = P(Y≤y) = 1- P(Y>y)

$$F(y) = 1 - \sum_{j=m+1}^{\infty} (1-p)^{2r} \sum_{k=j}^{\infty} \binom{r+k-1}{k} \binom{r+k-j-1}{k-j} p^{2k-j} \qquad (3.2)$$

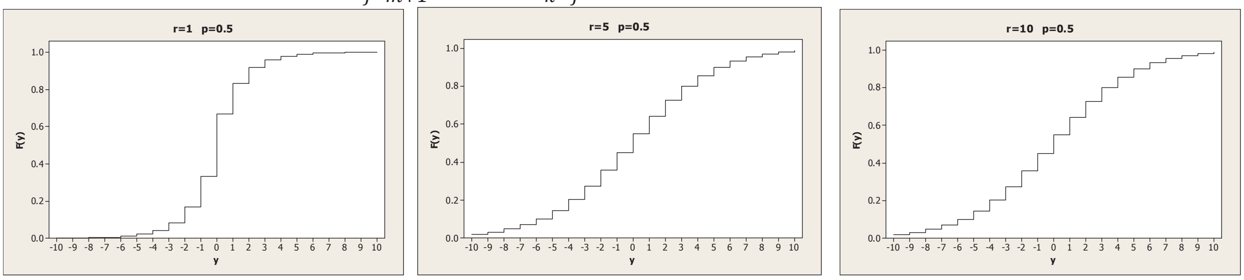

Fig. 2. Distribution function plot for different values of r and p=0.5.

The Moment generating function

The moment generating function of discrete bilateral gamma distribution is

$$M_Y(t) = E(e^{ty}) = \left(\frac{(1-p)^2}{(1-pe^{t})(1-pe^{-t})} \right)^r, -logp < t < log\,p \qquad (3.3)$$

Various moments are

$$E(Y) = \mu_1' = 0,$$

$$E(Y^2) = \mu_2' = \frac{2rp}{(1-p)^2}$$

$$E(Y^3) = \mu_3' = 0$$

$$E(Y^4) = \mu_4' = \frac{2rp}{(1-p)^2} + \frac{12r(r+1)p^2}{(1-p)^4}$$

Here all odd order moments are zero and closed form expressions are available for all even order moments .

$Mean = \mu_1' = 0$ and $Variance = \mu_2' - {\mu_1'}^2 = \frac{2rp}{(1-p)^2}$

Table 1. Variance of DBG(r,p) distribution for different values of r and p.

r\p	.1	.25	.5	.75	.9
.1	.02469	0.0888	.4	2.4	18
1	0.2469	0.8888	4	24	180
5	1.2346	4.4444	20	120	900
10	2469.0	8.8889	40	240	1800
20	4.9380	17.778	80	480	3600

$$Pearson's\ coefficient\ of\ Skewnes = \frac{\mu_3^2}{\mu_2^3} = 0$$

$$Pearson's\ coefficient\ of\ kurtosis = \frac{\mu_4}{\mu_2^2} = \frac{1}{r}\left[\frac{p}{2}\left(\frac{1}{p} - 1\right)^2 + 3(r+1)\right]$$

which is always greater than 3. Hence DBG(r,p) is always symmetric and leptokurtic.

Factorial Moments

The first four factorial moments of DBG (r,p) is given by

$$F_1 = E(Y) = 0$$

$$F_2 = E(Y)(Y-1) = \frac{2rp}{(1-p)^2}$$

$$F_3 = E(Y)(Y-1)(Y-2) = \frac{-6rp}{(1-p)^2}$$

$$F_4 = \frac{24rp}{(1-p)^2} + \frac{12r(r+1)p^2}{(1-p)^4}$$

The absolute moments are obtained as

$$E(|Y|)^n = 2(1-p)^{2r} \sum_{j=1}^{\infty} j^n \sum_{k=j}^{\infty} \binom{r+k-1}{k} \binom{r+k-j-1}{k-j} p^{2k-j}$$

Infinite Divisibility

A characteristic function $\varphi(t)$ satisfies the relation, $\varphi(t) = \{\varphi_n(t)\}^n$, for any n=1, 2, 3,... is said to be an infinite divisible characteristic function, where $\phi_n(t)$ are characteristic functions of any other random variable. Here (2.5) can be written as

$$\varphi_{y(t)} = \left[\left(\frac{(1-p)^2}{(1-pe^{it})(1-pe^{-it})}\right)^{\frac{r}{n}}\right]^n \quad (3.4)$$

where $\varphi_n(t) = \left(\frac{(1-p)^2}{(1-pe^{it})(1-pe^{-it})}\right)^{\frac{r}{n}}$ is the characteristic function of the difference of two negative binomial r.v's with parameters $\frac{r}{n}$ and q. Since negative binomial random variable is infinite divisible, then by the representation Y= X_1 -X_2 , DBG(r,p) is also infinite divisible.

Convolution Property

Let X_1 ,X_2...X_n be independent and i.i.d DBG(r,p) variables. Then consider the characteristic function of $Y = X_1 + X_2$

$$\varphi_Y(t) = \varphi_{X1}(t)\varphi_{X2}(t)$$

$$\varphi_Y(t) = \left[\frac{(1-p)^2}{(1-pe^{it})(1-pe^{-it})}\right]^{2r}$$

Hence Y follows DBG(2r,p). Also S_n= X_1+X_2+....+X_n follows DBG(nr,p).

Now consider $Z = X_1 - X_2$

$$\varphi_Z(t) = \varphi_{X1}(t)\varphi_{X2}(-t)$$

$$\varphi_Z(t) = \left[\frac{(1-p)^2}{(1-pe^{it})(1-pe^{-it})}\right]^{2r}$$

Hence Z follows DBG(2r,p). Hence the sum and difference of two DBG(r,p) variables again follows the same distribution.

Estimation : For the discrete bilateral gamma distribution with parameters r and p, the estimates can be obtained by the method of moments. Moment estimator for the parameters r and p are obtained by equating population moments with the corresponding sample moments.

$$r = -\frac{3A^2}{3A^2+An-Bn}, \quad where \quad A = \sum_{i=1}^{n} x_i^2 \;, \; B = \sum_{i=1}^{n} x_i^4 \quad (4.1)$$

$$p = \frac{3A^2 - 2An - Bn + \sqrt{3(A^2n^2+2ABn^2-6nA^3)}}{3A^2+An-Bn} \quad where \; 0 \le p < 1 \quad (4.2)$$

Empirical Study

In this section we fit discrete bilateral gamma distribution for the data obtained by random number generation. First we generate (1000 numbers) two independently and identically distributed negative binomial random variables say X_1 and X_2 with probability of success, q=0.7 Discrete bilateral gamma random variable Y is obtained as the difference of negative binomial random variables say Y = X_1-X_2.

The parameters of the model namely, r and p are obtained by the method of moments given by (4.1) and (4.2). Using the data we get r = 13.46046 and p = 0.253162. To test H_0: The data follows discrete bilateral gamma distribution, the following table gives the observed frequencies and expected frequencies of the data.

Table 2. Observed and expected frequency of the empirical data.

Y=X_1 -X_2	≥9	8	7	6	5	4	3	2	1	0	-1	-2	-3	-4	-5	- 6	-7	-8	≤-9	Total
Observed frequency	10	11	13	31	38	63	65	95	119	113	131	84	79	68	36	13	16	4	9	1000
Expected frequency	9	8	15	24	38	57	78	98	113	119	114	98	78	57	38	24	15	8	9	1000

$\chi^2 = 21.32$, Kolmogorov - Smirnov statistic D = 0.018

Thus based on Kolmogrov-Smirnov statistic and Chi-square test statistic we conclude that the null hypothesis of discrete bilateral gamma distribution as a good fit to the data is acceptable.

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