LIFE TESTING USING PROBABILITY DISTRIBUTIONS

PRATIBHA CHOUDHARY, ARUN JHAJHARIA

Abstract : This paper deals with the life data analysis using Weibull distribution and exponential distribution and parameter of interest i.e. failure rates, mean lifetime of the system have been obtained. Basis of parameter the distribution have been compared at the last the hypothetical examples have been given to highlight the results.

Keywords : About Accelerated life testing, Reliability, Weibull distribution, Exponential distribution, Mean life system.

Introduction : "Accelerated life testing" involves acceleration of failures with the single purpose of the "quantification of the life characteristics of the product at normal use conditions. This paper is solely concerned with this type of accelerated life testing.

Accelerated tests are becoming increasingly popular in today's industry due to the need for obtaining life data quickly. Life testing of product under higher stress levels without introducing additional failure modes can provide significant savings of both time and money. Correct analysis of data gathered via such accelerated life testing will yield parameters and information for the product's life under use stress conditions. Stress has been taken as random variable. The Weibull and exponential distribution have been studied. Such machine failure rates follow these distributions.

Methodology :

Continuous Probability Distribution:

A random variable X is said to be continuous if it can take all possible values (integral as well as fractional) between certain limits. In other words, a random variable is said to be continuous when its different values cannot be put in 1-1 correspondence with a set of positive integers.

Likelihood function

Let x_1, x_2,...x_n be a random sample of size n from a population with density function f(x,θ), then the likelihood function of the sample values x_1,x_2,...x_n, usually denoted by L= L(θ) is their joint density function, given by;

$$L = f(X_1\theta)f(X_2\theta)...f(X_n\theta) = \prod_{i=1}^{n} f(x_i,\theta)$$

Reliability

We shall define "the reliability of products" as the probability that it will function within specified limits for at least a specified period of time under specified environmental conditions.

That the component will fail between times t and t + Δt is given by f (t).Δt. Then the probability that the component will all on the interval from 0 to t is given by;

$$F(t) = \int_0^t f(x)dx$$

And the reliability function, expressing the probability that it survives to time t, is given by;

r(t) = 1- F(t)

The reliability is assessed to be no worse than some specific bound with a certain confidences. The confidence level determines the probability that the reliability bound contains the true reliability.

Result And Discussion:

Failure Rate Function for the Exponential Distribution: Consider a continuous random variable X having distribution function F and density f. The failure rate function r (t) is defined by;

Suppose now that the lifetime distribution is exponential. Then, by the memory-less property, it follows that the distribution of remaining life for a t-year-old item is the same as for a new item. Hence r(t) should be constant. This check out since

$$r(t) = \frac{f(t)}{1-F(t)}$$

$$= \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}} = \lambda$$

Thus, the failure rate function for the exponential distribution is constant. The parameter λ is often referred to as the rate of the distribution.

Weibull Distribution:

Weibull distribution adequately describes the failure times of components when their failure rate either increases or decreases with time. It has the parameters α and β its formula is given by;

Weibull distribution

$$f(t) = \alpha\beta t^{B-1} e^{-\alpha\beta}$$

t >0 , α >0 , $\beta > 0$

Weibull reliability function

R (t) = $e^{-\alpha t^{\beta}}$

The failure rate leading to the Weibull distribution is given by;
Weibull failure rate function

Z (t) = $\alpha\beta\ t^{\beta-1}$

Thus the Weibull distribution is IR when $\alpha \geq 1$, and DFR when 0 < $\alpha \leq$ 1; when α =1, G (t) = $1-e^{-\lambda t}$, the exponential distribution, which is both increasing Failure.
The MLE of Exponential Distribution:
The probability density function of exponential distribution is;

f(x) = $\lambda e^{-\lambda t}$

These methods are discussed in the literature. Some of the important references are: Bai (2003), Singh (2004), Balakrishnan (2007), Gajjar and Patel (2008), Han (2008).

The likelihood function for λ, given an independent and identically distributed sample x = (x_1, x_2... x_n) drawn from the variable is;

$$L(\lambda)=\prod_{i=1}^{n}\lambda e^{(-\lambda x_i)}=\lambda^n e^{\left(-\lambda\sum_{i=1}^{n}x_i\right)}=\lambda^n e^{(-\lambda n\bar{x})}$$

Where, $\bar{x}=\frac{1}{n}\sum_{i=1}^{n}x_i$ is the sample mean.
The derivative of the likelihood function's logarithm is;

$$\frac{\partial}{\partial\lambda}\ln L(\lambda)=\frac{\partial}{\partial\lambda}n\ln(\lambda)-\lambda n\bar{x})=\frac{n}{\lambda}-n\bar{x}\begin{cases}>0\,if\ 0<\lambda<\frac{1}{\bar{x}}\\ =0\,if\lambda=\frac{1}{\bar{x}}\\ <0\,if\lambda=\frac{1}{\bar{x}}\end{cases}$$

Consequently the maximum likelihood estimate for the rate parameter is;

$$\hat{\lambda}=\frac{1}{\bar{x}}$$

While this estimate is the most likely reconstruction of the parameter λ, it is only an estimate, and as such, one can imagine that the more data points are available the better the estimate will be. It's happened that one can compute an exact confidence interval- that is, a confidence interval that is valid for all number of samples, not just large ones. The 100(1-α)% exact confidence interval for this estimate is given by;

$$\frac{1}{\hat{\lambda}}\frac{2n}{X^2_{2n;\alpha/2}}<\frac{1}{\lambda}<\frac{1}{\hat{\lambda}}\frac{2n}{X^2_{2n;1-\alpha/2}}$$

The MLE of Weibull Distribution

Weibull distribution density function is given by;

$$f(x)=\frac{\beta}{\eta}\left(\frac{x-\gamma}{\eta}\right)^{\beta-1}e^{-\left(\frac{x-\gamma}{\eta}\right)}$$

β > 0, η > 0, x ≥ γ ≥ 0 ... (1)

These methods are discussed in the literature. Some of the important references are: Berrettoni (1964), Ahmad (2002), Abdel-Ghani (1998, Ghitany (2005), Saleh and Agerwal (2006).

Let x_1, x_2 ...x_n, be a random sample of size n drawn from a probability density function fx (x:θ) where θ is an unknown parameter. The likelihood function of this random sample is the joint density of the n random variable and is a function of the unknown parameter.

$$L=\prod_{i=1}^{n}f_{xi}(x_i\theta) \quad \text{... (2)}$$

In the likelihood function, The maximum likelihood estimator (MLE) of θ, Say $\hat{\theta}$ is the value of θ that maximizes L or equivalently, the logarithm of L, often, but not always, the MLE of θ is a solution of;

$$\frac{\partial\log L}{\partial\theta}=0,\quad \frac{\partial^2\log L}{\partial\theta^2}<0$$

Where solutions that are not functions of the sample values x_1, x_2,... x_n are not admissible, nor are solutions which are not in the parameter space, now we are going to apply the MLE to estimate the Weibull parameter, namely the shape and the scale parameter, consider the Weibull pdf given in (1) then likelihood function will be;

$$L(x_1..x_n:\beta,\eta)=\prod_{i=1}^{n}\left(\frac{\beta}{\eta}\right)\left(\frac{x_i}{\eta}\right)^{\beta-1}e^{-\left(\frac{x_i}{\eta}\right)} \quad \text{... (3)}$$

On taking the logarithms of (3), differentiating with respect to β and η in turn and equating to zero, we obtain the estimating equations;

$$\frac{\partial\ln L}{\partial\beta}=\frac{n}{\beta}+\sum_{i=1}^{n}x_i-\frac{1}{\eta}\sum_{i=1}^{n}x_i^{\beta}\ln x_i=0 \quad \text{... (4)}$$

$$\frac{\partial\ln L}{\partial\eta}=-\frac{n}{\eta}+\frac{1}{\eta^2}\sum_{i=1}^{n}x_i^{\beta}=0 \quad \text{... (5)}$$

On eliminating η between these two equations and simplifying, we have

$$\frac{\sum_{i=1}^{n}x_i^{\beta}\ln x_i}{\sum_{i=1}^{n}x_i^{\beta}}-\frac{1}{\beta}-\frac{1}{n}\sum_{i=1}^{n}x_i=0 \quad \text{... (6)}$$

Which may be solved to get the estimate of $\mu_x=\beta$. This can be accomplished by the use of standard iterative procedures (i.e. Newton- Raphson method). Once β is determined, η can be estimated using equation (5) as;

$$\eta = \frac{\sum_{i=1}^{n} x_i^{\beta}}{n}$$

The Mean life of Exponential distribution: $f(x,\lambda) = \lambda e^{-\lambda x}, x \geq 0$

The mean life of exponential distribution (Huang and Chan (2002)).

Then,

$$M_x(t) = E(e^{tx})$$

$$M_x(t) = \lambda \int_0^{\infty} e^{tx} e^{-\lambda x} dx$$

$$M_x(t) = \lambda \int_0^{\infty} e^{\{-(\lambda - t)x\}} dx$$

$$M_x(t) = \frac{\lambda}{\lambda - t} = \left(1 - \frac{t}{\lambda}\right)^{-1}$$

$$M_x(t) = \sum_{r=0}^{\infty} \left(\frac{t}{\lambda}\right)^r, \lambda > \lambda$$

$\mu_r' = E(x^r) =$ coefficient of $\frac{t^r}{r} in$

$$M_x(t) = \frac{r!}{\lambda^r} \quad \text{r= 1, 2, ...n}$$

Mean $\mu_1 = \frac{1}{\lambda}$

Weibull Distribution

$$f(x:c,\alpha,\mu) = \frac{c}{\alpha}\left(\frac{x-u}{\alpha}\right)^{c-1} \exp\left\{-\left(\frac{x-u}{\alpha}\right)^{c}\right\} : x >$$

For standard Weibull distribution (α =1, μ =0) we get $\gamma = X^c$

Which has the exponential distribution, we have;

$$\mu'_r = E(X^r)$$

$$\mu'_r = E(\gamma^{1/c})^r$$

$$\mu'_r = E(\gamma^{r/c})$$

$$\mu'_r = \int_0^{\infty} e^{-y} y^{r/c} dy$$

$$\mu'_r = \tau\left(\frac{r}{c} + 1\right)$$

Mean = $E(X) = \tau\left(\frac{1}{c} + 1\right)$

Increasing Failure Rate (IFR) and Decreasing Failure Rate (DFR).

Example1: The two basic formulas for the reliability of series and parallel system can be used in combination to calculate the reliability of a system having both series and parallel parts. To illustrate such a calculation, consider the system diagrammed in figure-1, which consists of eight components having the reliability shown in that figure. Find the reliability of the system.

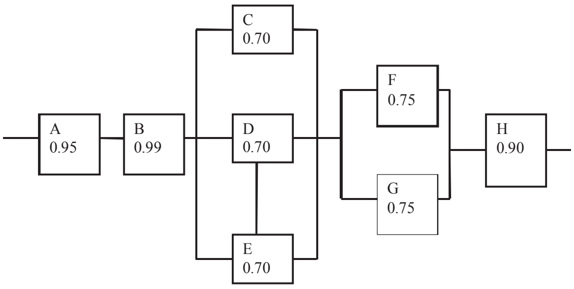

Fig. 1 Reliability components

The parallel assembly C, D, E can be replaced by a equivalent component C' having the reliability $1-(1-0.70)^3 = 0.973$, without affecting the overall reliability of the system, Similarly, the parallel assembly F, G can be replaced by a single component F' having the reliability $1-(1-0.75)^2 = 0.9375$. The resulting series A, B, C', F', H, equivalent to the original system, has the reliability.

Product law of reliability $R_s = \prod_{i=1}^{n} R_i$

= (0.95) (0.99) (0.973) (0.9375)(0.90)

= 0.772

Example 2 : Suppose that 50 units are replaced on life test (without replacement) and that the test is to be truncated r =10 of them have failed. We shall suppose, furthermore, that the first 10 failure times are 65, 110, 380, 420, 505, 580, 650, 840, 910 and 950 hours, Estimate the mean life of the component, its failure rate, and calculate a 0.90 confidence interval for μ.

Using the data of the preceding example test whether the failure rate is 0.23 failures per thousand against the alternative that the failure rate is less. Use the 0.05 level of significance.

Since n= 50, r=10,

$$T_{10} = (65 + 110 + ... + 950) + (50 - 10)950$$

= 43410 hours

And estimate the mean life of the component as $\hat{\mu} = \frac{43410}{10} =$ 4341 hours. The failure rate α is estimated by $\frac{1}{\hat{\mu}} = 0.00023$ failure per hours, or 0.23

failures per thousand hours. Also, a 0.90 confidence interval for u is given by;

$$\frac{2T_r}{x_2^{\ 2}} < \mu < \frac{2T_r}{x_1^{\ 2}}$$

$$\frac{2(43,410)}{31.410} < \mu < \frac{2(43,410)}{10.851}$$

$$0r$$

$$2764 < \mu < 8001$$

(1) Null hypothesis: $.\ \mu = \frac{1000}{0.23} = 4347.83 hours$

Alternative hypothesis: $\mu > 4347.83 hours$

(2) Level of significance: $\alpha = 0.05$

Criterion: Reject the null hypothesis if $T_r > \frac{1}{2}\mu_0 X^2{}_{0.05}$ where $X^2{}_{0.05} = 31.410$ is the value of $X^2{}_{0.05}$ having 20 degrees of freedom.

Calculations: Substituting r=10 and $\mu_0 = 4347.83$, we find the critical value for this test to be

$$\frac{1}{2}\mu_0 X^2{}_{0.05} = \frac{1}{2}(4347.83)(31.410) = 68282.67 \quad (5)$$

Decision: Since T_{10} =43,410 exceeds the critical value, we must reject the null hypothesis, concluding that the mean life-time exceeds 4347.83 hours, or equivalently, that the failure rate is less than 0.23 failure per thousand hours.

Example 3: One hundred devices are put on life test and the times to failure (in hours) of the first 10 that fail are as follows 7.0, 14.1, 18.9, 31.6, 52.8, 80.0, 164.5, 355.4, 451.0 and 795.1.

Assuming a Weibull time distribution estimates the parameters α and β as well as the failure rate at 1,000 hours. How does this value of the failure rate compare with the value of the failure rate compare with the value we would obtain if we assume the exponential model?

Suppose that a sample of 100 components is put on life test for 1000 hours and that the timed to failure of the 10 components that failed during the test are as follows: 7.0, 14.1, 18.9, 31.6, 52.8, 80.0, 164.5, 355.4, 451.0 and 795.1 hours.

In Weibull distribution

$$F(\hat{t_i}) = \frac{i}{n+1}$$

$$x_i = Int_i$$

$$y = InIn\frac{1}{1-F(\hat{t_i})}$$

We obtain Table 1.

$F(t_i)$	t_i	x_i	Y_i
0.010	7.0	1.94	-4.61
0.020	14.1	2.64	-3.91
0.030	18.9	2.93	-3.50
0.040	31.6	3.45	-3.21
0.050	52.8	3.96	-2.98
0.059	80.0	4.38	-2.79
0.069	164.5	5.10	-2.63
0.079	355.4	5.87	-2.49
0.089	451.0	6.11	-2.37
0.099	795.1	6.67	-2.26

$$\frac{\sum_{i=1}^{r} t_i^{\beta} Int_i + (n-r)t_r^{\beta} Int_r}{\sum_{i=1}^{r} t_i^{\beta} + (n-r)t_r^{\beta}} - \frac{1}{\beta} - \frac{1}{r}\sum_{i=1}^{r} \text{int}_i = 0.....(1)$$ Put

$\hat{\beta} = 0.7148$ in equation (1)

Then equation is 0=0, by this we get the value of z

$$\alpha = \frac{1}{\frac{1}{r}\left[\sum_{i=1}^{r} t_i^{\beta} + (n-r)t_r^{\beta}\right]}$$

$$\alpha = \frac{1}{\frac{323.36}{10}}$$

α =0.0309

Mean time to failure Weibull model

$$\mu = \alpha^{-1/\beta}\Gamma\left(1+\frac{1}{\beta}\right)$$

$$\mu = 0.0309^{-1/0.7148} * 0.69(1+\frac{1}{0.7148})$$

$$\mu = 0.0309^{-1/0.7148} * 0.69(1+1.39)$$

$\mu = 0.25 * 0.69 * 2.39$

$\mu = 0.412$

Weibull failure rate function $Z(t) = \alpha\beta t^{\beta-1}$

$Z(t) = 0.0309 * 0.7148 t^{0.7148-1}$

$Z(t) = 0.022 * t^{0.7148-1}$

$t = 1$

$Z(t) = 0.022$

In exponential distribution :

$T_{10} = (7.0 + 14.1 + 18.9 + ... + 795.1) + (100 - 10)795.1$

= 735294

and we estimate the mean life of the component as

$$\hat{\mu} = \frac{735294}{100} = 7352.94$$

Confidence interval for mean

$$\frac{2T_r}{x_2^{\ 2}} = \frac{2T_r}{x_1^{\ 2}} \qquad \frac{2(735294)}{x_2^{\ 2}} < \mu < \frac{2(735294)}{x_1^{\ 2}}$$

$$\frac{2(735294)}{31.410} < \mu < \frac{2(735294)}{10.851}$$

$$46819.1 < \mu < 135525.5$$

The failure rate α is estimated by

$$\alpha = \frac{1}{\hat{\mu}}$$

$$\alpha = \frac{1}{7352.94}$$

$$\alpha = 0.00036$$

The failure rate of exponential distribution is less then Weibull distribution failure rate, then we uses the exponential distribution.

References

1. Ahmad I. A., On moment inequalities of the supremum of empirical processes with applications to kernel estimation.
2. Statist. Probab. Lett. 57, 215-220.
3. Al-Saleh, J. and Agarwal, S. Extended Weibull type distribution and finite mixture of distributions. Statistical Methodology, 3, 2006, 224-233.
4. Bai J. and Perron P., Critical values for multiple structural change tests, The Econometrics Journal, 6(1), 2003, 72–78.
5. Balakrishnan, N and Sarhan, A., A new class of bivariate distribution and its mixture", Journal of Multivariate Analysis, vol. 98, 2007, 1508-1527.
6. Berrettoni, J. Practical applications of the Weibull distribution. Industrial Quality Control, 21, 1964, 71-79.
7. Cohen, A. C., Progressively Censored Samples in Life Testing. Technometrics, 5, 1998, 327-329.
8. Gajjar, K.A. and Patel, M.N. (2008). Estimation for a Mixture of Exponential Distribution based on Progressively Type –II Censored Sample.
9. Ghaly, A.A., Attia, A.F. and Abdel- Ghani, M.M.. The Estimation Problem of Partially Accelerated Life Tests for Weibull distribution by Maximum Likelihood method with Censored Data, Proceeding of the 31st Annual Conference of Statistics, Computer Science and Operation, ISSR, Cairo University, 1996, 128-138.
10. Ghitany, M., Al-Hussaini, E. and Al- Jarallah, R. Marshall-Olkin extended Weibull distribution and its application to censored data. J. App. Statist., 32, 2005,1025-1034.
11. Han, Y., T. J. Wang, and R. Z. Rao, Progress in the study of physic-optics characteristics of atmospheric aerosol, Acta Phys. Sinica (in Chinese), 57(11), 2008,7396–7407.
12. Singh, P., Simultaneous Confidence Intervals for the Successive Ratios of Scale Parameters., J Stat Plan Infer., 36(3): 2004, 1007-1019.

***

New Housing Board Colony M/100, Prem Nagar, Balaghat- 481001 (M.P.)

Department of Genetics and Plant Breeding, College of Agriculture, Rewa Jawaharlal Nehru Krishi Vishwa Vidyalaya, Jabalpur- (M. P.), India

Email: prati.gudiya1989@gmail.com , Mobile No. 07898880326

D-1-214, Vallabh Garden, Rajmata Sudarsana Nagar, Bikaner- 334001, Rajasthan

Department of Farm Engineering, Institute of Agricultural Sciences,Banaras Hindu University, Varanasi221005

Email: arunstatistics@gmail.com, Mobile No. 08765120512