ON TOPOLOGY OF THE FRACTAL SPACE

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Abstract : In this paper, we discuss the topology induced by Hausdorff metric on the set of all non empty compact subsets of a complete metric space. We examine the relation between this topology and the convergence of sets. We also prove the convergence of a sequence of sets which are attractors to a converging sequence of Iterated Function Systems.

Keywords : Convergence, Iterated Function System, Sequence, Topology.

Introduction: Here we discuss the ideal space to study fractal geometry. We work in a complete metric space (X,d) and then consider the set of all compact subsets of *X* with Hausdorff metric. Informally, the Hausdorff metric, named after Felix Hausdorff, measures the largest length out of the set of all distances between each point of a set to the closest point of a second set. Given a metric space, the Hausdorff metric induces a topology on the space of all compact subsets of the metric space. This space is referred to as the "space of fractals" or the fractal space. We investigate some interesting properties of this topology in the first section.

Iterated function systems (IFS) are one of the most common and general ways to generate fractals. The notion of IFS was introduced by Hutchinson [5] and was popularised by Barnsley [1]. These methods are useful tools to build fractals and other self similar sets as attractors of dynamical systems. Fractal methods are quite popular in the modelling of natural phenomena in computer graphics, engineering sciences, physics and so forth.

We discuss some results on the convergence of the attractors of the IFS in the next section.

Hausdorff Metric Topology: Let (X,d) be a complete metric space. Consider K(X), the set of all non-empty compact subsets of X. We define a metric h on K(X) as follows:

$$h(A,B) = \max\{\sup_{x\in A}\{dist(x,B)\}, \sup_{y\in B}\{dist(y,B)\}\}$$

for all subsets $A,B \in$ K(X). Here h is called the Hausdorff metric on K(X).

Consider the topology on K = K(X) U ϕ induced by the Hausdorff metric h called the Hausdorff metric topology.

The collection of all ε-balls $B_h(A,\varepsilon)$ for $A \in$ K and $\varepsilon > 0$, forms a basis for the Hausdorff metric topology T_h. Here $B_h(A,\varepsilon) = \{B / h(A, B < \varepsilon)\}$.

The metric h preserves the metric on X. ie; $h(\{x\},\{y\}) = d(x,y)$ for all $x, y \in X$.

Analogous to the definitions and results from the classical analysis, we make the following definitions.

Definition 2.1: A sequence $(A_1, A_2, \ldots\ldots)$ of subsets of K(X) is said to converge to a compact set *A* iff for every neighbourhood {*U*} of A, there exist a positive integer N such that $A_i \in \{U\}$, for all $i \geq N$ and we say $\{A_n\}$ converges to A, ie; $A_n \to A$.

Definition 2.2: A sequence $(A_1, A_2, \ldots\ldots)$ of subsets of K(X) is said to be a Cauchy sequence in K(X), if there exist a positive integer N such that for all $m,n > N, h(A_m, A_n) < \varepsilon$, for a given $\varepsilon > 0$.

Definition 2.3: The Hausdorff metric space (K, h) is complete if every Cauchy sequence $\{A_n\}$ is complete in K.

Theorem 2.4 If the metric space (X,d) is complete, (K, h) is complete.

Proof can be found in Barnsley [1].

Theorem 2.5 If (X,d) is totally bounded, (K, h) is totally bounded.

Proof: Fix $\varepsilon > 0$.

We have X totally bounded. Let F be a finite subset of X with $X \subset B_h(F,\varepsilon)$.

For each compact subset *A* of *X*, there exist a minimal subset E of F such that $A \subset B_h(E,\varepsilon)$

And it follows that the reverse inclusion $E \subset B_h(A,\varepsilon)$ must be satisfied. As a result, each compact subset of *X* has Hausdorff distance at most ε from some subset of F. Since F has only finitely many subsets and $\varepsilon > 0$ was arbitrary, (K, h) is totally bounded.

Theorem 2.6 If (X,d) is a compact metric space, (K, h) is compact.

Proof: A metric space is compact iff it is complete and totally bounded. Hence the theorem follows from Theorem 2.2 and 2.3.

Similar proofs on the set of all closed subsets of *X* can be found in Beer [2] and Michael [6].

Convergence Of Sequences: In this section, we discuss the convergence of sequence of sets in the Hausdorff distance. Before going to the main results, we will see some of the basic definitions and results from Falconer [3], [4] required for the understanding of the main results.

Definition 3.1: A mapping $f : X \to X$ is a contraction if there is a number c with $0 < c < 1$: $d(f(x), f(y)) \le c.d(x, y)$ for all $x, y \in X$. Clearly, every contraction is a continuous mapping.

Definition 3.2 Let $\{f_n\}$ be a sequence of contractions and c_n is the contraction factor of f_n for each $n \in N$. Then the contraction factor of the sequence is defined as $c = \max\{c_n, n \in N\}$.

Definition 3.3: A hyperbolic iterated function system (IFS) consists of a complete metric space together with a finite set of contraction mappings $f_j : X \to X$ with respect to the contraction factors $c_j, j = 1,..., N$. The contraction factor of the IFS is defined as $c = \max\{c_j, j = 1,..., N\}$. The notation for the IFS just defined is $\{X; f_j, j = 1,..., N\}$.

Then the system $(f_1,...., f_N)$ generates a natural contraction mapping F on K(X) with contraction factor c: $F(B) = \bigcup_{i=1}^{N} f_i(B)$.

By Banach fixed point theorem, there exist a unique set $A \in$ K(X) such that $F(A) = A$, and for any $B \in$ K(X), the sequence of iterates $F^n(B)$ converges to A. The unique fixed point A is called the attractor of the IFS $\{X; f_j, j = 1,..., N\}$.

Theorem 3.4 Let (X, d) be a complete metric space and $\{f_n\}$ be a sequence of contractions on X with contraction factor 'c' which is pointwise convergent to a contraction f with contraction factor 'c'. Let x_n be the fixed point of f_n and x be the fixed point of f. If $\{x_n\}$ is bounded. Then $x_n \to x$.

Proof: For any $n \in N$, we have

$$d(x_n, x) \le d(f_n(x_n), f_n(x)) + d(f_n(x), f(x))$$

$$\le c.d(x_n, x) + d(f_n(x), f(x))$$

Since $\{f_n\}$ converges to f, the result follows.

The following two theorems are from Mihail [7] which will be used to prove the main results.

Theorem 3.5 Continuity Theorem: Let (X, d) be a complete metric space, f and g are contractions on X. let α be the fixed point of f and β be the fixed point of g. Then

$$d(\alpha, \beta) \le \overline{d}(f, g)\frac{1}{1 - \min\{Lipf, Lipg\}}$$ where $\overline{d}(f, g) = \sup_{x \in X} d(f(x), g(x))$ and

$$Lipf = \sup_{x,y \in X, x \ne y} \frac{d(f(x), f(y))}{d(x, y)}.$$

Theorem 3.6 Let (X, d) be a complete metric space and $x \in X$. Let $I_1 = \{X; f_i, i = 1,..., n\}$ and $I_2 = \{X; g_i, i = 1,..., n\}$ be two IFSs with contraction factors c_1 and c_2 and attractors A_1 and A_2 respectively.

Then $h(A_1, A_2) \le \max_{k=1,..,n} \overline{d}(f_k, g_k).\frac{1}{1 - \min.(c_1, c_2)}$,

where $\overline{d}(f_k, g_k) = \sup_{x \in X} d(f_k(x), g_k(x))$.

Main Results

Theorem 3.1.1: Let (X, d) be a complete metric space. Let $\{X; f_1,..., f_m\}$ be an IFS with attractor $A \in$ K(X) such that $A = \bigcup_{i=1}^{m} f_i(A)$. If there exist a sequence of contractions $\{f_{nk}\}$ which converges to $f_k, k = 1,..,m.$ Then $\{X; f_{j1},..., f_{jm}\}$ is an IFS having attractor , A_j for $j = 1,2,....$ and the sequence of sets $\{A_n\}$ converges to A.

Proof: We have a sequence $\{f_{nk}\}$ converging to $f_k, k = 1,..,m.$, and an IFS $\{X; f_1,..., f_m\}$ with attractor A.

Since $\{f_{j1},..., f_{jm}\}$ for each j is a finite set of contractions with contraction factor c_j, the IFS $\{X; f_{j1},..., f_{jm}\}$ has an attractor A_j. Hence we get a sequence $\{A_n\}$ of sets in K(X).

Using Thoerem 3.6, for all i, j

$$h(A_i, A_j) \le \max_{k=1,..,m} \overline{d}(f_{ik}, f_{jk}).\frac{1}{1 - \min.(c_i, c_j)}.$$

But $\overline{d}(f_{ik}, f_{jk})$ decreases, since $\{f_{nk}\}$ converges to $f_k, k = 1,..,m.$, and $\{c_n\}$ converges to, say c, the contraction factor of the IFS $\{X; f_1,..., f_m\}$.

Hence,

$h(A_n, A) \le \max_{k=1,..,m} \bar{d}(f_{nk}, f).\dfrac{1}{1-\min.(c_n, c)} \to 0$ as $n \to \infty$.

in the case of above theorem, if there exist converging sequences of contraction mappings to each member of an IFS on X, we get a converging sequence of contraction mappings on K(X). This result is summarised in the following theorem.

Theorem 3.1.2

Let (X, d) be a complete metric space. Let $\{X; f_1,...,f_m\}$ be an IFS with attractor $A \in$ K(X) such that $A = \bigcup_{i=1}^{m} f_i(A)$. If there exist a sequence of contractions $\{f_{nk}\}$ which converges to f_k, for $k = 1,..,m$. Then there exist a sequence of contraction mappings $\{F_n\}$ on K(X) which converges to F whose fixed point is A.

ie; $\{A_n\}$ converges to A on K(X).

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Proof: For each j, $\{f_{j1},..., f_{jm}\}$ is a finite set of contractions which induces a natural mapping F_j on K(X) such that $F_j(B) = \bigcup_{i=1}^{N} f_{ji}(B)$.

Since F_j is a contraction mapping on a complete metric space, it has a unique fixed point A_j. Now using theorem 3.1.1, this sequence $\{A_n\}$ converges to A, which is the fixed point of a contraction mapping, say F, induced by $\{f_1,..., f_m\}$ the collection of contraction mappings on X. Thus the sequence $\{F_n\}$ converges to F on K (X).

Conclusion: We have discussed some of the properties of Hausdorff metric topology, and proved some results on the convergence of a sequence of compact subsets of a complete metric space. We wish to extend our study of the convergence in the case of infinite IFS.

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