AN M/M/(S,S) QUEUING – INVENTORY SYSTEM WITH EXPONENTIAL LEAD TIME AND LOST SALES

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Abstract : In this paper, we consider a multi server queuing model with the special feature that the servers act as an inventory with standard (s,S) policy and positive lead time. The demands are considered as arrivals and the time for serving the inventory is considered as the service time. With each replenishment, the level of the inventory is restocked to S (<∞) . The lead time follows an exponential distribution. We assume that during stock out, new arrivals do not join the queue and customers being in the system, wait until their service is completed. Using the Matrix Geometric method, behavior of this system is analyzed and a numerical illustration is given.

Keywords : Matrix analytic methods, (s,S) inventory policy.

Introduction : Inventory management system at a service facility has been extensively analyzed by many researchers such as Sigman and Levi (1992), Berman et al.(1993), Schwarz et al.(2006), Krishnamurthy and Islam(2004). In most of the models, an M/M/1 queue is formed where the server serves an inventory. The server uses one item of the inventory for each service provided. Various inventories such as production inventories, inventories with/without lead time, perishable inventory etc. attached with a queue are examined with different ordering policies such as (r,Q),(s,S) and other general randomized policies. One can refer Krishnamoorthy et al.(2011) for further details. A multi server queuing system which serves an inventory have also been investigated recently. An M/M/c queue attached with a continuous review perishable inventory system is analyzed by Yadavally et al.(2011).

An M/M/(s,S) queuing inventory model introduced by Anoop et al.(2012) is quite different from the usual multi server inventory models. Here the inventory is considered as the servers with standard (s,S) policy and the time for serving an inventory is the service time. When a customer leaves the system after completing service, one server also leaves the system which results in the decrement of the total number of servers by one. They have investigated M/M/(s,S) model without lead time and analyzed the system using a two state Quasi Birth and Death (QBD) process using matrix analytic methods. In an earlier work, we have investigated the system with exponential lead time without lost sales(Anoop (2012)). In this paper we consider an M/M/(s,S) queuing inventory system with positive exponential lead time and lost sales during stock out.

The rest of the paper is organized as follows. Section 2 describes the model and a stability condition for the existence of the process is derived in section 3. Section 4 gives an algorithm to find the steady state probabilities and section 5 describes the various performance measures. A numerical illustration for the model is given in section 6.

Model Description : We consider a multi server Markovian queuing model where the servers are considered as an inventory with standard (s,S) policy. We assume that the arrival process is Poisson with rate λ. When a customer leave the system after service completion, one server also leaves the system which results in the decrement of the total number of servers by one. When the number of servers is reduced to s , an order is immediately placed. With each replenishment the inventory is restocked to 'S', no matter how many items are still present in the inventory. The replenishment lead time is exponential with parameter α>0. The time for serving an inventory is assumed to be exponential with parameter μ. When the inventory level is zero, an arriving customer does not join the queue and all the customers in the queue will wait until their service is completed.

The state of the system is described by means of a bivariate process Ω={N(t) , M(t) } where N(t) represents the number of customers in the system at time t and M(t) is the inventory size at time t. Then Ω is a continuous time Markov chain with state space {0,1,2,...}×{0,1,2,...,S} which turns out to be a QBD process. The infinitesimal generator Δ of the process Ω is of the form

$$\Delta = \begin{pmatrix} A_1^{(0)} & A_0^{(0)} & & & & \\ A_2^{(1)} & A_1^{(1)} & A_0^{(1)} & & & \\ & A_2^{(2)} & A_1^{(2)} & A_0^{(2)} & & \\ & & \ldots\ldots & \ldots\ldots. & \ldots\ldots. & \\ & & A_2^{(S)} & A_1^{(S)} & A_0^{(S)} & \\ & & & A_2^{(S)} & A_1^{(S)} & A_0^{(S)} \\ & & & \ldots\ldots & \ldots.. & \end{pmatrix}$$

Where the block matrices $A_i^{(n)}$ are described as follows.

When n=0,1,2,...S

$A_0^{(n)} = \left(a_{uv}^{(n)}\right)$,

$a_{uv}^{(n)} = \begin{cases} \lambda, if\ u = v \neq 0 \\ 0, \quad otherwise \end{cases}$ where u,v = 0,1,...,S

$A_1^{(n)} = \left(b_{rt}^{(n)}\right)$,

$$b_{rt}^{(n)} = \begin{cases} -(\lambda + min(r,n)\,\mu + \alpha), if\ r = t\ and\ r \leq s \\ -(\lambda + min(r,n)\,\mu),\ if\ r = t\ and\ r > s \\ -\alpha, \quad if\ r = t = 0 \\ \alpha, \quad if\ t = s\ and\ r \leq s \\ 0, \quad otherwise \end{cases}$$

where r,t = 0,1,...,S and

$A_2^{(n)} = \left(c_{pq}^{(n)}\right)$,

$c_{pq}^{(n)} = \begin{cases} min(p,n)\,\mu,\ if\ p = q+1 \\ o, \quad otherwise \end{cases}$ where p,q =0,1,...,S

It is clear from the definition of $A_0^{(n)}, A_1^{(n)}$ *and* $A_2^{(n)}$ that the QBD process is level dependant for 0≤ n <S and level independent for n≥S.

Stability Condition: In this section we discuss the necessary and sufficient condition for the stability of the QBD described above. Define $A = A_2^{(S)} + A_1^{(S)} + A_0^{(S)}$. Let Θ=(θ₀, θ₁, θ₂,..., θ_S) be the steady state probabilities for the generator matrix A. The steady state distribution Θ can be obtained from the equation ΘA=0 and Θe=1. Thus we get the following equations.

$$-(n\mu + \alpha)\theta_n + (n+1)\mu\theta_{n+1} = 0 \quad .....(1)$$

for n=0,1,2,...,s

$$-(n\mu)\theta_n + (n+1)\mu\theta_{n+1} = 0 \quad .....(2)$$

for n=s+1,s+2,...,S

$$\alpha(\theta_0 + \theta_1 + \cdots + \theta_s) - S\mu\theta_S = 0 \quad .....(3)$$

And the normalizing equation,

$$\theta_0 + \theta_1 + \cdots + \theta_S = 1 \quad .....(4)$$

Thus we have S+2 equations in S+1 variables. (Note that equation (3) is a linear combination of equations (1) and (2))

Solving these system of equations, we get the steady state distribution of the generator matrix A as,

$$\theta_i = \left(\prod_{j=0}^{i-1} \frac{(\alpha + j\mu)}{(j+1)\mu}\right), \text{for i=1,2,...,s+1} \quad ....(5)$$

$$\theta_i = \frac{s+1}{i}\theta_{s+1}, \text{ for i=s+2,s+3,...,S} \quad .....(6)$$

Where θ_0 is given by the equation ,

$$\theta_0 = \left[\left(1 + \sum_{k=1}^{s}\prod_{j=0}^{k-1}\frac{(\alpha + j\mu)}{(j+1)\mu}\right) + \left(\prod_{j=0}^{s}\frac{(\alpha + j\mu)}{(j+1)\mu}\right)\left((s+1)\sum_{j=s+1}^{S}\frac{1}{j}\right)\right]^{-1} \quad .....(7)$$

It is well known that the standard drift condition $\Theta A_0^{(S)} e < \Theta A_2^{(S)} e$ (refer Neuts [7]) is necessary and sufficient for the stability of the QBD process. That is,

$$(\theta_0, \theta_1, \ldots, \theta_S)A_0^{(S)}e < (\theta_0, \theta_1, \ldots, \theta_S)A_2^{(S)}e \quad .....(8)$$

Substituting $\theta_j, A_0^{(S)}$ *and* $A_2^{(S)}$ we get the condition,

$$\lambda(\theta_1 + \theta_2 + \theta_3 + \cdots + \theta_S) < \mu(\theta_1 + 2\theta_2 + 3\theta_3 + \cdots + S\theta_S) \quad .....(9)$$

That is ,

$$\lambda < \mu\frac{(\theta_1 + 2\theta_2 + 3\theta_3 + \cdots + S\theta_S)}{(\theta_1 + \theta_2 + \theta_3 + \cdots + \theta_S)} \quad ...(10)$$

θ_j's are given by equations (5),(6) and (7).

The QBD process with generator matrix Δ is stable if and only if $\lambda < \mu\frac{(\theta_1 + 2\theta_2 + 3\theta_3 + \cdots + S\theta_S)}{(\theta_1 + \theta_2 + \theta_3 + \cdots + \theta_S)}$

Steady State Solution : Here we find the steady state solution of the QBD process described above. Let $\Pi = (\pi_0, \pi_1, \pi_2, \ldots)$ be the steady state probabilities of the process with infinitesimal generator Δ where each π_n is an (S+1) dimensional vector . That is $\pi_n = (\pi(n,0), \pi(n,1), \ldots, \pi(n,S))$ for n= 0,1,2,... The QBD process is state independent for n>S . Therefore the steady state solution is of the form

$$\pi_{S+n} = \pi_S R^n \text{ for n=1,2,3,... } .....(11)$$

where R is the solution of the matrix quadratic equation

$$R^2 A_2^{(S)} + R A_1^{(S)} + A_0^{(S)} = 0 \text{ (refer Neuts [7]) } .....(12)$$

We use the following algorithm to find the steady state probabilities π_n for n=0,1,2,...,S

From the equation $\Pi\Delta = 0$, we get

$$\pi_{S-1}A_0^{(S-1)} + \pi_S A_1^{(S)} + \pi_{S+1}A_2^{(S+1)} = 0 \quad .....(13)$$

$$\text{Thus,}\ \pi_S = \pi_{S-1}T_S, \quad .....(14)$$

$$where\ T_S = -A_0^{(S-1)}(A_1^{(S)} + RA_2^{(S)})^{-1}$$

Proceeding like this we get the following recursive relations ,

$$\pi_j = \pi_{j-1}T_j, for\ j = S-1, S-2, \ldots, 1 \quad .....(15)$$

$$where\ T_j = -A_0^{(j-1)}(A_1^{(j)} + T_{j+1}A_2^{(j+1)})^{-1}$$

Now π_0 can be solved from the system of equations

$$\pi_0 = \pi_0 P\ and\ \pi_0 Qe = 1 \quad .....(16)$$

where P and Q are given by

$$P = \left(-T_1 A_2^{(1)}\right)(A_1^{(0)})^{-1} \quad \text{and}$$

$$Q = 1 + \sum_{i=1}^{S-1}\prod_{k \leq i} T_k + \left(\prod_{k=1}^{S} T_k\right)(I - R)^{-1}$$

Hence the steady state distribution of the entire system can be calculated using equations (11), (14),(15) and (16).

System Performance Measures: Now we are in a position to calculate the performance measures of the system. We present below some of the important measures.

1) Average number of customers in the system

$$E_{sys} = \sum_{n\geq 0}\sum_{m=0}^{S} n\,\pi(n,m)$$

2) Expected number of servers in the system

$$E_{ser} = \sum_{n\geq 0}\sum_{m=0}^{S} m\,\pi(n,m)$$

3) Average number of customers in the queue

$$E_{que} = \sum_{n>m}\sum_{m=0}^{S}(n-m)\,\pi(n,m)$$

4) Probability that an arriving customer has to wait $P_{wait} = \sum_{n\geq m}\sum_{m=1}^{S}\pi(n,m)$

5) Probability that an arriving customer does

not join the queue $P_{lost} = \sum_{n=0}^{\infty} \pi(n,0)$

6) Effective arrival rate $\lambda_{eff} = \lambda(1 - P_{lost})$

7) Average waiting time of a customer in the system $W_{sys} = \frac{E_{sys}}{\lambda_{eff}}$

8) Average waiting time of a customer in the queue $W_{que} = \frac{E_{que}}{\lambda_{eff}}$

6. Numerical Illustration : In this section, we provide numerical illustrations of the system performance. We consider the following example.

Let the model be M/M/(2,4) and the service rate μ=3 and lead time rate α=2 . Using the stability condition (10) we get λ ≤ 6. The steady state probabilities are given in Table 1 for 0≤n≤10 when λ=4

Table (2) shows various performance measures of the system M/M/(s,4) when λ=4, μ=3 and α=2 for different values of s .

Table (2): Performance Measures

Values of s	E_{sys}	E_{que}	E_{ser}	P_{lost}
0	4.65929	3.33726	2.18579	0.495760
1	2.81361	1.53009	2.37063	0.302916
2	2.34832	1.06143	2.58392	0.234718
3	2.17737	0.88403	2.75677	0.20419

Table1: Steady state probabilities for M/M/(2,4)

Number of customers	Number of servers: 0	1	2	3	4
0	0.065189	0.030656	0.043759	0.064485	0.069802
1	0.056554	0.043459	0.061312	0.087518	0.08598
2	0.027445	0.02744	0.027582	0.032418	0.030876
3	0.017427	0.018297	0.016024	0.016886	0.014188
4	0.010754	0.011618	0.009152	0.009107	0.007487
5	0.006523	0.007169	0.00523	0.005081	0.004237
6	0.003917	0.004349	0.003008	0.002905	0.002469
7	0.002337	0.002611	0.001744	0.001687	0.001454
8	0.00139	0.001558	0.001018	0.000988	0.000859
9	0.000825	0.000927	0.000597	0.000582	0.000508
10	0.000489	0.00055	0.000351	0.000343	0.000301

Conclusion : In this paper, we have analyzed a queuing inventory model with positive lead time and lost sales. It can be seen that there is a strong dependance between the number of customers and number of servers in the system. The work on M/M/(s,S) system with retrial of customers and other variants are in progress.

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