MODULATION THEOREM EXPLAINS THE BEHAVIOR OF THE FCST

N.M. SHARMA, S.S. MATHURKAR, J.S. WATH, AND DR. P.A. GULHANE

Abstract : FCST is useful tool not only in mathematics but also in optic signal processing and many other areas of science and engineering. In this paper we have proved the modulation theorem and some important results about the kernel of transform we have extended the result for Heaviside's unit step function based on same transformation as it is used in structural mechanics to describe different types of structural tools.

Keywords: Fourier Cosine Stieltjes transform (FCST), Modulation theorem, and Heaviside's unit step function.

Introduction The Fourier Transform provides frequency resolution of a (possibly non-periodic) function. For finite energy signals, the Fourier Transform will exist and any band limited signals will have Fourier Transform with a compact region of support. Fourier Transform breaks up the function into a continuous spectrum of frequencies s.

The Fourier Cosine Transform (which is real) of $f(t)$ in the interval $0<t<\infty$ is defined as follows

$$F_c(s)=\int_0^\infty f(t)\cos st dt \qquad (1.1)$$

For bounded signals (with possibly infinite energy), alternate definition of band limitation are possible by using the Fourier-Stieltjes Transform [1].

The conventional Fourier Cosine Stieltjes Transform of a complex valued smooth function $f(t,x)$ is defined by the convergent integral.

$$F(s,y) = FS\{f(t,x)\}$$

$$= \int_0^\infty \int_0^\infty f(t,x) cos\ st (x+y)^{-p} dxdt \qquad (1.2)$$

The unit step function also called Heaviside's unit step function $U(t)$, is defined as

$$U(t) = \begin{cases} 0, & t<0 \\ 1, & t \geq 0 \end{cases} \qquad (1.3)$$

The unit step function is a curve which has value zero at all points to the left of the origin and is equal to one on the right of the origin.

The notation and terminology will follow that Zemanian [2].In section 2, testing function space $FCS_{\alpha,\alpha}(\Omega)$ are defined. Section 3, proves the Modulation Theorem. Properties on Fourier Cosine Stieltjes (FCS) transform and properties related to Heaviside's unit step function on FCS transform are defined in section 4 and section 5, In section 6, concludes the paper.

Testing Function Space $FCS_{a,\alpha}(\Omega)$:

A function $\Psi(t,x)$ defined on $0<t<\infty, 0<x<\infty$ is said to be a member of $FCS_{a,\alpha}$ if $\Psi(t,x)$ is smooth and for each non negative integer l,q

$$\gamma^{a,k,l,q}_{(\Psi)} = Sup_{0<t,x<\infty}|t^k(1+x)^a D_t^l (xD_x)^q \psi(t,x)|$$
$$\leq C_{l,q}A^a a^{a^\alpha}, \quad a = 0,1,2$$

Where the constants A and $C_{l,q}$ depend on the testing Ψ. For $k=0$, we get $a^{a^\alpha}=1$, the topology of the space $FCS_{a,\alpha}$ is the space generated by the countable multinorm, $S=\cup_{l,k=0}^\infty\{\gamma_{a,k,l,q}\}$. With this topology $FCS_{a,\alpha}$ is a countably multinorm space. A sequence $\{\Psi_v\}$ is said to converge in $FCS_{a,\alpha}$ to Ψ if for each non negative integer l,q we have, $\gamma_{a,k,l,q}(\Psi_v-\Psi)\to 0$ as $v\to\infty$. We define distributional FCS transform of any function f in dual space $FCS_{a,b}$ i.e. $FCS^*_{a,b}$ by,

FCS {f(t,x)}=F(S,Y)

$$= \langle f(t,x), cos\ st\ (x+y)^{-p}\rangle$$

For complex parameters s and y .The R.H.S. has sense for $f \in FCS^*_{a,b}$ and $\cos st(x+y)^{-p} \in .FCS_{a,b}$

Modulation Theorem:

Statement: - If $FCS(s,y)$ is the Fourier Cosine Stieltjes transform of the function $f(x,t)$ then

$$FCS\{f(x,t)cosat\} = \frac{1}{2}\{[FCS(s+a,y)] + [FCS(s-a,y)]\}$$

Proof:- $FCS\{f(x,t)cosat\} =$

$$\int_0^\infty \int_0^\infty f(x,t) cos\ at\ cos\ st\ (x+y)^{-p} dxdt$$

$$= \frac{1}{2}\int_0^\infty \int_0^\infty f(x,t)\ cos[t(a+s)]\ (x+y)^{-p} dxdt$$

$$+\frac{1}{2}\int_0^\infty \int_0^\infty f(x,t)\ cos[t(a-s)]\ (x+y)^{-p} dxdt$$

$$= 1/2\ \{[FCS(s+a,y)] + [FCS(s-a,y)]\}$$

Properties on Fourier Cosine Stieltjes (FCS) transform:

4.1. (a) $\langle f+g,\ \varphi\rangle = \langle f,\ \varphi\rangle + \langle g,\ \varphi\rangle$,

(b) $\langle \alpha f,\ \varphi\rangle = \langle f,\ \alpha\,\varphi\rangle$

Proof: (a) $\langle f+g,\varphi\rangle =$

$$\int_0^\infty\int_0^\infty (f+g)(x,t)\cos st(x+y)^{-p}\,dxdt$$

$$= \int_0^\infty\int_0^\infty f(t,x)\cos st(x+y)^{-p}dxdt\ +$$

$$\int_0^\infty\int_0^\infty g(t,x)\cos st(x+y)^{-p}dxdt$$

$= \langle f,\ \varphi\rangle + \langle g,\ \varphi\rangle$,

Proof: (b) This proof is similar and hence omitted.

4.2. If f is a distribution and ψ is a function which is infinitely smooth then

$$\langle \frac{\partial}{\partial t}(\psi f),\varphi\rangle = \langle \psi\frac{\partial f}{\partial t},\varphi\rangle + \langle f\frac{\partial \psi}{\partial t},\varphi\rangle$$

Proof: $\langle \frac{\partial}{\partial t}(\psi f),\varphi\rangle$

$$=\int_0^\infty\int_0^\infty \frac{\partial}{\partial t}[(\varphi f)(x,t)]\cos st\,(x+y)^{-p}dxdt$$

$$= \int_0^\infty\int_0^\infty \left[\varphi(x,t)\frac{\partial f(x,t)}{\partial t} + f(x,t)\frac{\partial \varphi(x,t)}{\partial t}\right]\cos st(x+y)^{-p}\,dxdt$$

$$= \langle \psi\frac{\partial f}{\partial t},\varphi\rangle + \langle f\frac{\partial \psi}{\partial t},\varphi\rangle$$

4.3 $FCS\{f'(x,t)\} = sFSS\{f(x,t)\} - f(0)$

Proof: $FCS\{f'(x,t)\}=$

$$= \int_0^\infty\int_0^\infty f'(x,t)\cos st(x+y)^{-p}dxdt$$

$$=\int_0^\infty\{\int_0^\infty f'(x,t)\cos st\,dt\,\}\,(x+y)^{-p}dx$$

$$= -\int_0^\infty f(x,0)\,(x+y)^{-p}dx + s\int_0^\infty\int_0^\infty f(x,t)\sin st\,(x+y)^{-p}dxdt$$

$$=sFSS\{f(x,t\} - f(0)$$

By using above properties we can prove the following results for Heaviside's unit step function on FCS transform

Properties Of Heaviside's Unit Step Function On FCS Transform:

$$\langle\delta(t-a),\frac{\cos st}{(s+t)^p}\rangle = \frac{\cos sa}{(s+a)^p} \quad (5.1)$$

$$\langle\delta(t),\frac{\cos st}{(s+t)^p}\rangle = \frac{1}{s^p} \quad (5.2)$$

$$\langle\delta(t-a)+\delta(t+a),\frac{\cos st}{(s+t)^p}\rangle = \frac{\cos sa}{(s+a)^p} + \frac{\cos sa}{(s-a)^p} \quad (5.3)$$

$$\boldsymbol{\delta * f = f} \quad (5.4)$$

$$\langle\delta(t-a)*f(t),\emptyset(t)\rangle = \langle f(t-a),\emptyset(t)\rangle \quad (5.5)$$

Proofs of (5.1) to (5.5) are simple and hence omitted.

Conclusion: In this paper we proved Modulation theorem on FCST and properties can be used to understand and evaluate FCST. A function is modulated by another function if they are multiplied in time. It uses in all most all branches of engineering, particularly in optics, radar, signal processing etc. For bounded signals (with possibly finite energy) alternate definition of band limitation are possible by using the FCS transformation.

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[1]Dept. Of Mathematics, H.V.P.M.'s College of Engg. & Technology, Amravati, (M.S), India
nidhimanish.sharma337@gmail.com
[2]Dept. of Mathematics, Govt. College of Engineering, Amravati, (M.S), India
ssmathgcoea@gmail.com
[3]Dept. of Mathematics, P.R.Pote College of Engg. & Technology, Amravati, (M.S), India
jotsnawath@gmail.com
Dept. Of Mathematics, Govt. College of Engineering, Amravati, (M.S), India
gulhane.padmaja@gcoea.ac.