STRONG DOMINATING χ-COLOR NUMBER OF SOME GRAPH

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Abstract: In this paper, we define strong and weak-dominating χ-color number of a graph G as the maximum number of color classes which are strong and weak dominating sets of G respectively, and are denoted by sd_χ and wd_χ respectively, where the maximum is taken over all χ-coloring of G. Also we discuss the conditions under which sd_χ is equal to the chromatic number χ. Further, the strong-dominating χ-color number for sum and Cartesian product of graphs are discussed.

Keywords: Dominating number, Chromatic number, Dominating χ-Color number, Central, Middle and Product graph.

Introduction: Let G = (V(G), E(G)) be a simple, connected, finite, undirected graph. The order and size of G are denoted by n and m [2]. In graph theory, coloring and dominating are two important areas which have been extensively studied. The fundamental parameter in the theory of graph coloring is the chromatic number $\chi$of a graph G which is defined to be the minimum number of colors required to color the vertices of G in such a way that no two adjacent vertices receive the same color. If $\chi(G)=k$, we say that G is k-chromatic.

A set $D \subseteq V(G)$ is a dominating set of G, if for every vertex $x \in V(G)\setminus D$ there is a vertex $y \in D$ with $xy \in E(G)$ and D is said to be strong dominating set of G, if it satisfy the additional condition $d(x,G) \le d(y,G)$[1]. The strong domination number $\gamma_{st}(G)$ is defined as the minimum cardinality of a strong dominating set. A set $S \subseteq V(G)$ is called weak dominating set of G if for every vertex $u \in V(G)\setminus S$, there exists vertex $v \in S$ such that $uv \in E(G)$ and $d(u,G) \ge d(v,G)$. The weak domination number $\gamma_w(G)$ is defined as the minimum cardinality of a weak dominating set and was introduced by Sampathkumar and PushpaLatha (Discrete Math. 161 (1996)235-242)[3].

In this paper, we list some important results related with coloring and domination in section 2. Also in section 3, we define a new parameter strong-dominating χ-color number sd_χ and prove the conditions under which sd_χ is equal to the chromatic number χ. Further, the strong-dominating χ-color number for sum and Cartesian product of graphs are discussed. In section 4, we give some open problem for future work.

Preliminary Result: Let G be a graph of order n whose n vertices are listed in some specified order. In greedy coloring of G, if the vertices of G are listed in the order $v_1,v_2,\dots,v_k$, then the resulting greedy coloring c assigns the color 1 to v_1. If v_2 is not adjacent to v_1 then assign the color 1 to v_2, otherwise assign the color 2. In general, suppose that first j vertices $v_1,v_2,\dots,v_j$, $1\le j < n$ in the sequence have been colored and t is the smallest positive integer not used in coloring any neighbor of v_{j+1} from among $v_1,v_2,\dots,v_j$. Then assign the color t to v_{j+1}. This algorithm is stated more formally, Suppose that C is a k-coloring of a graph G, where each color is one of the integers 1,2,…,k as mentioned above. If V_i $(1\le i \le k)$ is the set of vertices in G colored i (where one or more of these sets may be empty), then each nonempty set V_i is called a color class and the nonempty elements of $\{V_1,V_2,\dots,V_k\}$ produce a partition of V(G). If $\chi(G)=k$, we say that G is k-chromatic.

Definition 2.1: [6] Let G be a graph with $\chi(G)=k$. Let $C=V_1,V_2,\dots,V_k$ be a k-coloring of G. Let d_C denote the number of color classes in C which are dominating sets of G. Then $d_\chi(G)= max_c\, d_C d_C$ where the maximum is taken over all the k-colorings of G, is called the dominating χ-color number of G.

Definition 2.2: [5] Let G be a graph with $\chi(G)=k$. Let $C= V_1,V_2,\dots,V_k$ be a k-coloring of G. Let d_C denote the number of color classes in C which are dominating sets of G. Then $md_\chi(G)= max_c\, d_C$where the minimum is taken over all the k-colorings of G, is called the min-dom-color number of G.

Definition 2.3: [4] Let G_1 and G_2 be two graphs. Cartesian product $G_1\times G_2$is the graph having vertex set $V(G_1\times G_2)=V(G_1)\times V(G_2)$ and edge set $E(G_1\times G_2)=\{(u_1,v_1)(u_2,v_2)/u_1u_2\in E(G_1)$ and $v_1 = v_2$ or $v_1v_2\in E(G_2)$ and $u_1 = u_2\}$.

Theorem 2.4 [Brook's]: [2]If G is connected graph other than complete graph or an odd cycle then $\chi(G) \le \Delta(G)$.

Theorem 2.5: [5] Let G be a χ-critical graph. Then $md_\chi(G)=\chi(G)$ if and only if G is complete.

Theorem 2.6: [5] Let G_1 and G_2 be two graphs. Then

$md_\chi(G_1+G_2)= md_\chi(G_1) + md_\chi(G_2)$

$d_\chi(G_1+G_2)= d_\chi(G_1) + d_\chi(G_2)$.

Theorem 2.7: [5] Let $G_1\times G_2$be the Cartesian product of two graphs G_1 and G_2.

If $\chi(G_2)<\chi(G_1)$, then $md_\chi(G_1\times G_2)= \chi(G_1\times G_2)$ if and only if $md_\chi(G_1)=\chi(G_1)$.

If $\chi(G_2)=\chi(G_1)$, then $md_\chi(G_1\times G_2)= \chi(G_1\times G_2)$ if and only if $md_\chi(G_1)=\chi(G_1)$ or $md_\chi(G_2)=\chi(G_2)$.

Main Results Arumugam et al. [7] discussed parallel processing system as the application of dominating χ-color number. There are many circumstances in real life where scheduling and resource allocation are needed to combine to find an optimum solution. In this paper, we try to find all the of possibility to schedule and allocate the resources with optimum condition like cost, timing, power, energy, profit, data transmission etc., For that, we use strong (weak) domination instead of domination and define a parameter strong(weak)-dominating χ-color number. To find the possibility of optimized results of the problem which include scheduling and allocating, we define the following definition.

Let G be a graph with χ(G)=k. Let C=$V_1, V_2, \ldots, V_k$ be a k-coloring of G. Let d_C denote the number of color classes in C which are strong (weak) dominating sets of G. Then $max_c\, d_C$ where the maximum is taken over all the k-colorings of G, is called the strong (weak)-dominating χ-color number of G. And strong and weak dominating χ-color number of G is denoted by $sd_\chi(G)$ and $wd_\chi(G)$ respectively.

Note: Strong-dominating χ-color number $sd_\chi(G)$ exists for all graphs G; $1 \le sd_\chi(G) \le d_\chi \le \chi(G)$.

Weak-dominating χ-color number $wd_\chi(G)$ exists for all graphs G; $1 \le wd_\chi(G) \le d_\chi \le \chi(G)$.

Theorem3.1 :For all integers m and n with 0< m < n-1, then there exist a graph G with n vertices and $sd_\chi(G)$=m. And m=n iff G is complete.

Proof. Case (i) If G is complete graphthen every color class has a vertex which strongly dominates all other vertex. Conversely, if m=nand assume G is not complete. Since some vertex do not dominate all other vertices, the dominating χ-color number is less then n. This is contradiction to our assumption.

Case (ii) m<n-1 that is to prove this by the construction of a graph G with $sd_\chi(G)$=m from K_n by deleting some edges. Let $\{v_1,v_2,\ldots,v_n\}$ be the vertices and $\{e_{i1},e_{i2},\ldots,e_{i(n-1)}\}$ be the edges incident with v_i. First, delete an edge e_{i1} from the graph G. Degree of two incident vertices, say v_i,v_j of e_i is decreased by one. Since number of n-1 degree vertices is n-2, the strong-dominatingχ-color number $sd_\chi(G)$ is n-2.

And then delete any edge whose one end incident with the vertices v_i and the another end incident with maximum degree vertex, say v_k. Since number of n-1 degree vertices are n-3, the strong-dominating χ-color number $sd_\chi(G)$ is n-3.

Similarly, continue this process until the graph has only one n-1 degree vertex. So, It is possible to delete n-2 edges whose one ends incident with the vertex v_i and another end incident with the vertices of maximum degree. Therefore, from this construction it is easy to get the graphs with n vertices whose strong dominating χ-color number is n-2,n-3,…,1.

Observations:

1. $\chi(K_n) = d_\chi(K_n) = md_\chi(K_n) = sd_\chi(K_n) = wd_\chi(K_n) = n$
2. $sd_\chi(C_n) = wd_\chi(C_n) = \begin{cases} 3 & if\ n = 3 \\ 2 & if\ n > 3 \end{cases}$
3. Let W_n be the wheel of order n, $sd_\chi(W_n) = \begin{cases} 4 & if\ n = 4 \\ 1 & if\ n > 4 \end{cases}$ and $wd_\chi(W_n) = \begin{cases} 4 & if\ n = 3 \\ 2 & if\ n > 4 \end{cases}$
4. $sd_\chi(K_{1,n}) = wd_\chi(K_{1,n}) = 1$ for n >1.
5. If G is any complete k-partite graph with partition C=$V_1, V_2, \ldots, V_k$, then
   a. $sd_\chi(G) = |X|$ where X={V_i | d(v,G=Δ(G), v∈ V_i}.
   b. $sd_\chi(G) = |Y|$ whereY={V_i | d(v,G=δ(G), v∈ V_i}.

Let s(G) and t(G) be denote minimum and maximum cardinality of any set in any χ-chromatic partition of G respectively, where minimum and maximum are taken over all possible χ-chromatic partitions of G.

Theorem 3.2: If G is any graph with s(G)=t(G), then $sd_\chi(G) = wd_\chi(G) = \chi(G)$

Proof.Let C be any χ-chromatic partitions of G and let D∈C.

If D is not a strong dominating set of G, then there exist a vertex v which is not dominated by D or d(u,G)<d(v,G). If D is not a weak dominating set of G, then there exist a vertex v which is not dominated by D or d(u,G)>d(v,G). It is now possible to get another χ-chromatic partition with the set D∪{v} having cardinality |D|+1. But this is impossible since s(G)=t(G) means each color class in any χ-chromatic partition has the same cardinality

Theorem 3.3: If G is connected graph with n vertices other than K_n or C_n, then $sd_\chi(G) \le \Delta(G)$ and $wd_\chi(G) \le \Delta(G)$.

Proof.By using Brook's theorem, and $0 \le sd_\chi(G) \le \chi(G)$, $0 \le wd_\chi(G) \le \chi(G)$, we can prove this theorem.

We now characterise graphs with $sd_\chi(G) = \chi(G)$.

Theorem3.4 : Let G be a χ-critical graph. Then $sd_\chi(G)=\chi(G)$ iff G is complete.

Proof.Using theorem 2.5 and definition of χ-critical graph, the result holds.

Theorem 3.5: Let $G_1 \times G_2$ be the Cartesian product of two graphs G_1 and G_2. If $\chi(G_1)>\chi(G_2)$, then $sd_\chi(G_1\times G_2)=\chi(G_1\times G_2)$ if and only if $sd_\chi(G_1)=\chi(G_1)$.

Proof.By theorem 2.7(1), If $\chi(G_2)<\chi(G_1)$, then $md_\chi(G_1\times G_2)= \chi(G_1\times G_2)$ if and only if $md_\chi(G_1) =\chi(G_1)$. So, it is enough to prove that every color class of $G_1\times G_2$ has maximum degree vertices if and only if every color class of G_1 has maximum degree vertices.

Let B and C be any χ-chromatic partitions of $G_1\times G_2$ and G_1 respectively. Also let D∈B and E∈C. Then our claim is there exist a vertex u∈D, d(u, $G_1\times G_2$)=Δ($G_1\times G_2$) if and only if there exist a vertex v∈ E, d(v,G_1) = Δ(G_1). Suppose every vertex v∈E, d(v,G_1) <Δ(G_1). In the construction of $G_1\times G_2$, there exist a D∈ B, such that every vertex u∈D, d(u, $G_1\times G_2$)<Δ($G_1\times G_2$).

This is contradiction. Similarly, Converse also holds.

Corollary 3.6: Let $G_1 \times G_2 \times \ldots \times G_n$ be the Cartesian product of n graphs $G_1, G_2, \ldots, G_n$. If $\chi(G_1) > \chi(G_2) > \ldots > \chi(G_n)$, then $sd_\chi(G_1 \times G_2 \times \ldots \times G_n) = \chi(G_1 \times G_2 \times \ldots \times G_n)$ if and only if $sd_\chi(G_1) = \chi(G_1)$.

Proof:It can be easily proved by taking induction on n, with the theorem 3.5.

Theorem 3.7: Let G_1 and G_2 be two graphs. The sum G_1+G_2 is the graph having vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$ together with all the edges joining the points of $V(G_1)$ to the points $V(G_2)$. Then

1. $sd_\chi(G_1+G_2) = sd_\chi(G_1) + sd_\chi(G_2)$ if $|V(G_1)| - \Delta(G_1) = |V(G_2)| - \Delta(G_2)$.
2. $sd_\chi(G_1+G_2) = sd_\chi(G_1)$ if $|V(G_1)| - \Delta(G_1) < |V(G_2)| - \Delta(G_2)$.

Proof. Let B be any χ_1-chromatic partitions of G_1 and C be any χ_2-chromatic partitions ofG_2. By definition of the sum graph of G_1 and G_2, G_1+G_2 has a $\chi_1 + \chi_2 -$ chromatic partition.

Let B_{sd} and C_{sd} be set of all strong dominating set of color classes of G_1 and G_2 respectively, such that $B_{sd} \subseteq B, C_{sd} \subseteq C$ and $B_{sd} \cap C_{sd} = \emptyset$.

1. Let D_{sd} be set of all strong dominating set of color classes of G_1+G_2.

Now our claim is $D_{sd} = B_{sd} \cup C_{sd}$. Let $D \in D_{sd}$ and If D is not a strong dominating set of $G_1 + G_2$, then there exist a vertex $v \in G_1+G_2$which is not dominated by D or $d(u,G_1+G_2) < d(v,G_1+G_2)$.Without loss of generality, let $D \epsilon B_{sd}$. Since every vertex of G_1 dominates all vertices of G_2 in G_1+G_2, D dominates vertices of G_2 in G_1+G_2.

So, $v \in G_1$.Therefore $D \notin B_{sd}$. This contradicts.

And supposed$(u,G_1+G_2) < d(v,G_1+G_2)$. If $v \in G_2$, then $\max d(u,G_1+G_2) < \max d(v,G_1+G_2) \Rightarrow \max d(u,G_1) + |V(G_2)| < \max d(v,G_2) + |V(G_1)| \Rightarrow \Delta(G_1) + |V(G_2)| < \Delta G_2) + |V(G_1)|$. This contradicts.

2. We shall proveB_{sd} is strong dominating set of different color classes of G_1+G_2. Let $E \in C_{sd}$ and If E is a strong dominating set of G_1+G_2, then there exists a vertex $y \in E$ such that $d(y,G_1+G_2) \geq d(x,G_1+G_2)$.So, $\max d(y,G_1+G_2) \geq \max d(x,G_1+G_2) \Rightarrow \max d(y,G_2) + |V(G_1)| < \max d(x,G_1) + |V(G_2)| \Rightarrow \Delta(G_2) + |V(G_1)| < \Delta(G_1) + |V(G_2)|$.

This contradicts.

Corollary 3.8 : For any two graphs G_1 and G_2 with $|V(G_1)| - \Delta(G_1) = |V(G_2)| - \Delta(G_2)$, $sd_\chi(G_1+G_2) = sd_\chi(G_1) + sd_\chi(G_2)$ if and only if$sd_\chi(G_1+G_2) = \chi(G_1+G_2)$ and $sd_\chi(G_2) = \chi(G_2)$.

Proof.Since$sd_\chi(G_1+G_2) = sd_\chi(G_1) + sd_\chi(G_2)$ and $\chi(G_1+G_2) = sd_\chi(G_1) + sd_\chi(G_2)$,

$\chi(G_1) + \chi(G_2) = sd_\chi(G_1) + sd_\chi(G_2)$.

$sd_\chi(G_1) < \chi(G_1) \Rightarrow sd_\chi(G_2) > \chi(G_2)$ or $sd_\chi(G_2) < \chi(G_2) \Rightarrow sd_\chi(G_1) > \chi(G_1)$

This is contradiction. Therefore,$sd_\chi(G_1) = \chi(G_1)$ and $sd_\chi(G_2) = \chi(G_2)$.

To prove sufficiency, $sd_\chi(G_1+G_2) = sd_\chi(G_1) + sd_\chi(G_2) = \chi(G_1) + \chi(G_2) = \chi(G_1+G_2)$.

Open Problems

1. Characterise the class of graph G with $sd_\chi(G) = 1$.
2. Characterise the class of graph G with $sd_\chi(G) = d_\chi(G)$.
3. Characterise the class of graph G with $sd_\chi(G) = md_\chi(G)$.

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