MULTIPLICATION SEMIGROUPS WITH IDENTITY

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Abstract: There are many aspects of semigroup study, analytical, topological, applied and algebraic. In this paper, algebraic aspect is discussed. The algebraic study of semigroups is developed in two ways (i) as generalization of groups and (ii) as generalization of rings. V.L.Mannepalli and Satyanarayana M., have used ring theoretic skills to study semigroups, one of which gave rise to the study of multiplication semigroups. This paper discusses the structure of commutative multiplicative semigroups containing identity element, having unique maximal ideal M. Here we also study the regular semigroups and principal ideal semigroups. Lastly we study the characterization of multiplication semigroups whose dimension ≤ 1, having unique maximal ideal M and whose every non maximal prime ideal is idempotent.

Keywords: Commutative semigroup, Regular semigroup, Radical of an ideal, Minimal prime divisor, Kernel of an ideal,

Introduction In this paper the structure of commutative multiplication semigroups containing identity element has been determined, along with some good results. This work has been developed by Mannepalli [8] and contains the following results.

In [3], authors have proved that every cancellative multiplication semigroup is a direct product of the additive semigroup of non-negative integers and an arbitrary abelian group. In general, regular semigroups and principal ideal semigroups containing identity elements form subclasses of multiplication semigroups. However, they do not exhaust all multiplication semigroups containing identity. Multiplication rings have been extensively studied by Gilmer and Mott [5], Griffin [6], Mott [11] etc. and have been characterized using powerful ideal-theoretic techniques. Their techniques are not enough to characterize multiplication semigroups, since multiplication semigroups are not necessarily of dimension less than or equal to one, unlike multiplication rings (the term 'dimension' is defined exactly the same way as in rings). For example, S= $\{1, x_1, x_2, \ldots\}$ with $x_i x_j = x_i$ for $i \leq j$ is a multiplication semigroups with infinite chain of prime ideals. Throughout this paper, (S,M) stands for a commutative semigroup S containing identity having a unique maximal ideal M. All semigroups under consideration are commutative semigroups which are not groups. All semigroups contain identity unless otherwise stated. For any ideal A, we denote $\bigcap_{i=l}^{\infty} A^i$ by A^w. Z denotes the set of all non-cancellable elements which is easily seen to be a prime ideal. The concepts and facts in this paper are analogous to those in commutative ring theory. For details the reader is referred to [7]. If P is a prime ideal containing an ideal A, using Zorn's lemma it is easy to verify that P contains a minimal prime divisor of A. For any two ideals A and B of S, we define A:B = $\{x \in S \mid xB \subseteq A\}$. For any prime ideal P of S, we define the semigroup of fractions Sp = { a/b | a ∈ S, b ∈ s\p is a cancellable element }. By defining a map η : S → S_ρ by η(s) = s/1 (s ∈ S), S can be identified with a subsemigroup of S_ρ. For any ideal A of S, the ideal generated by η(A) in Sρ is denoted by AS_ρ. $AS_\rho \cap S$ is an ideal of S; moreover if A ⊆ P then $AS_\rho \cap S$ is P-primary in S. For the undefined terms used in this paper, the reader is referred to [3].

Result

1.2.1. Proposition : Let S be a commutative semigroup with identity and having a unique maximal ideal M, then the following are true.

(i) If P is a prime ideal and A is any ideal such that P < A then P = PA; and P = A^w or P = PA^w.

(ii) Every ideal equaling to with its prime radical is a primary ideal; in particular, if P is a prime ideal, then P^n is a P-primary ideal for any +ve integer n.

(iii) If P is a prime ideal with $P^n \neq P^{n+1}$ for any +ve integer n, then P^w is a prime ideal.

(iv) Every primary ideal is a power of its radical.

(v) If P is a proper prime ideal and A is any ideal such that $A \subseteq P^n$, $A \nsubseteq P^{n+1}$ for some positive integer n, then P^n = A: yS for some y ∈ S \ P.

(vi) If A is an idempotent ideal, then A is a multiplication semigroup not necessarily containing identity. In particular, for any x ∈ A we have x = xy for some y ∈ A.

(vii) Every homomorphic image of S is a multiplication semigroup.

(viii) If dim S ≤ 1, then A = Ker A for every ideal A.

(ix) If P is a non-idempotent prime ideal, then there is at most one prime ideal Q < P with the property that there are no prime ideals between P and Q.

(x) If prime ideals are linearly ordered, then every non-idempotent prime ideal P is principal.

Proof:

(i) Since for ideals P and A; P < A and S is a multiplication semigroup, hence $P = AB$ for some ideal B of S. By hypothesis $A \not\subseteq P$ and since P is prime ideal, we have $P = AP = A(AP) = A^2P$ etc. so that $P \subseteq A^w$. If $P \neq A^w$ then it follows from the above that $P = PA^w$.

(ii) Let Q be any ideal with $\sqrt{Q} = P$, where P is a prime ideal. We have to prove that Q is primary. If P=S, trivially Q is primary. We assume $P \neq S$. Now suppose Q is not primary. Then for some $p \in P \setminus Q$ and $r \in S \setminus P$ we can have $pr \in Q$. Let $A = Q \cup (pS)P$. We note that $p \notin A$. For otherwise $p = pp'$ for some $p' \in P$. Now $p'^n \in Q$ for some n, so that we have $p = pp'^n \in Q$, which is a contradiction. Since $A \cup pS \subseteq P$, $A \cup pS = PL$ for some ideal L; and $P < P \cup rS$ implies $P = P(P \cup rS)$ by (i). Therefore, it follows that $A \cup pS = P(P \cup rS)L = (A \cup pS)(P \cup rS) \subseteq A$; i. e. $p \in A$, which is a contradiction. Hence Q is a primary ideal. Consequently, if P is a prime ideal P^n is P-primary for any positive integer n.

(iii) Let $x, y \in P^w$. If $x, y \notin P$, clearly $xy \notin P^w$. So it suffices to consider the case when $x \notin P$ and $y \in P$. Then $y \in P^r \setminus P^{r+1}$ for some positive integer r. Now $xy \notin P^{r+1}$ since P^{r+1} is P-primary by (ii). Thus $xy \notin P^w$. Finally, if $x, y \in P$, then $x \in P^r \setminus P^{r+1}$ and $y \in P^s \setminus P^{s+1}$ for some positive integers r and s. Now $xS = P^r A$ and $yS = P^s B$ with ideals $A, B \not\subseteq P$. If $xy \in P^{r+s+1}$, then $xyS = P^{r+s} AB \subseteq P^{r+s+1}$. But P^{r+s+1} is P-primary by (ii). So that $P^{r+s} \subseteq P^{r+s+1}$ since $AB \not\subseteq P$. Thus $P^{r+s} = P^{r+s+1}$ which is not true. Hence $xy \notin P^{r+s+1}$ i.e. $xy \notin p^w$. Thus p^w is a prime ideal.

(iv) Let Q be a primary ideal with $\sqrt{Q} = P$ we have to show that $Q = P^n$ for some positive integer n. If P = S, clearly Q = S and we are through. So assume that $P \neq S$. Suppose $Q \subseteq P^w$. If $P^n \neq P^{n+1}$ for any n, then by (iii), P^w is a prime ideal. This implies $P = \sqrt{Q} \subseteq \sqrt{P^w} = P^w$ which is a contradiction. Hence we have the following possibilities : either $Q \subseteq P^w$ with $P^n = P^{n+1}$ for some positive integer n, or $Q \subseteq P^n$, $Q \not\subseteq P^{n+1}$ for some positive integer n. In the first case, for any $x \in P^n$ we have $xS = P^n A$ for some ideal A. Then $xS = P^n A = P^{n+1} A = PP^n A = P(xS)$, so that x=tx for some $t \in P$. Since $\sqrt{Q} = P$, we have $t^k \in Q$ for some positive integer k. We now have $x = t^k x \in Q$. Hence $Q = P^n$. In the second case, we have $Q = P^n C$ where $C \not\subseteq P$. Since Q is P-primary, clearly $P^n \subseteq Q$ and hence $Q = P^n$.

(v) By hypothesis, we have $A = P^n B$ with $B \not\subseteq P$. Let $y \in B \setminus P$, then $P^n yS \subseteq A$, so then $P^n \subseteq A{:}yS$. If $x \in A{:}\,yS$, then $xy \in A \subseteq P^n$. Since P is prime ideal, P^n is P-primary by (ii), so that $x \in P^n$ Since $y \notin P$. Thus $A : y^s \subseteq P^n$ Hence the result.

(vi) If B is any ideal of A, then BS = AC for some ideal C of S. Now since A is an idempotent ideal of S, it is easy to verify that B is an ideal of S. Hence B = AD for some ideal D of S, which in turn implies B = AB. Let B_1 and B_2 be any two ideals of A with $B_1 \subseteq B_2$. Then B_1 and B_2 are also ideals of S, so that $B_1 = B_2L$ for some ideals L of S. Now $B_1 = B_2AL$ where AL is clearly an ideal of A. Hence A is a multiplication semigroup. If $x \in A$ then $xS = (xS)A = xA$ so that $x = xy$ for some $y \in A$.

(vii) Since S is a multiplication semigroup so for ideals A and B such that $A \subseteq B$, there exists an ideal C of S satisfying the condition A = BC. Let $\;: S \rightarrow T$ be a homomorphism. Clearly ϕ (A), ϕ (B) and ϕ (C) are ideals of T. It is easy to check that ϕ (A) and ϕ (B) are ideals of T such that $\phi(A) \subseteq \phi(B)$. Further there exists an ideal ϕ (C) satisfying the condition ϕ (A) = (B).ϕ(C) is a routine verification. Thus T is a multiplication semigroup.

(viii) Let A be any ideal. Suppose $A \neq$ Ker A. Let $a \in$ Ker A\A. Set B = A:aS. Let P be a minimal prime divisor of B. Clearly $A \subseteq B \subseteq P$, so that P contains a minimal prime divisor Q of A. Suppose Q = P. Now $AS_Q \cap S$ is an ideal of S with radical Q and thus is a Q-primary ideal in view of (ii), so that $a \in AS_Q \cap S$. Now we can write a/1 = b/x where $b \in A$, $x \in S \setminus Q$. Then $ax = b \in A$, from which we obtain $x \in A{:}aS = B \subseteq Q$ which is a contradiction. Therefore Q < P. Now if dim S<1, we must have Q = P and thus A = Ker A. If dim S=1, then P=M. By (ii) and (iv), $B=M^k$ for some k. Since $a \in Q$, we have aS = QC for some ideal c; more over $Q = QM^k$ by (i). Thus $aS = M^k(aS) = B(aS) \subseteq A$, i.e. $a \in A$, which is a contradiction. Hence A = Ker A.

(ix) Suppose Q and Q' are prime ideals such that Q < P and Q' <P. Assume that there are no prime ideals between Q and P, and Q' and P. Clearly $Q \cup Q' = P$; moreover Q = PQ and Q' = PQ' by (i). Now it follows that $Q \cup Q' = P^2$, i.e. $P=P^2$ which leads to a contradiction (P being a non-idempotent prime ideal). Hence the result.

(x) Let P be a non-idempotent prime ideal. We have to show that P is principal. Let $x \in P \setminus P^2$. Since prime ideals are linearly ordered, P is the only minimal prime divisor of xS. By (ii) P^n is a P-primary ideal for some positive integer n and by (iv) a P-primary ideal is a power of its radical, namely P. Hence $xS = P^r = P$. Thus P is a principal ideal.

1.2.2. Proposition: If (S, M) is a multiplication semigroup, then the following are true.

(i) For every ideal A of S either $A=M^n$ for some non-negative integer n or $A \subseteq M^w$; in particular, if $M^w = \phi$ then the set of ideals of S is dually well-ordered.

(ii) If x and y are cancellable and non-cancellable elements respectively then $yS < xS$; consequently $Z \lessdot xS$.

(iii) S/Z is a Dedekind semi-group (with zero adjoined if $Z \neq \phi$).

(iv) If $M \neq Z$, then $M = xS$ for some cancellable

element x; and $Z = M^w$.

(v) $Z = Z^2$ or $Z = yS$ for some non-idempotent element y.

Proof: (i) follows from (v) of proposition 1.2.1, if we set P = M, S/Z is a multiplication semigroup by (vii) of proposition 1.2.1. Now (ii), (iii) and (iv) follow from a well-known result and from the following fact of [4], viz : If S is a multiplication semigroup then $S = P \cup Z$, $P \cap Z = \phi$ where P is the set of all cancellable elements of S and $Z < aS$ for all $a \in P$. Now we prove (v) ; suppose $Z \neq Z^2$. If Z = M then Z = yS for any $y \in Z \setminus Z^2$, in view of (ii) and (iv) of proposition 1.2.1. Let $Z \neq M$. For any $y \in Z \setminus Z^2$, $yS = ZA$ where $A \not\subseteq Z = M^w$. Now from (i) we have $A = M^n$ for some non-negative integer n. Thus $yS = ZM^n = Z$, since $Z < M$ implies $Z = ZM^k$ for any non-negative integer k, by (i) of proposition 1.2.1

1.2.3. Corollary: Let (S,M) be a semigroup with $M^w = \phi$. Then S is a multiplication semigroup iff ideals of S are powers of M. Furthermore, if $M \neq Z$ then the multiplication semigroup S is a Dedekind semigroup.

1.2.4. Theorem : if S is a regular semigroup or a principal ideal semigroup, then S is a multiplication semigroup.

Proof : Consider any pair of ideals A and B with $A \subseteq B$. If S is a regular semigroup, one can easily check that A=AB. Since the ideals A and B can be written in the form $A = U e_{\propto} S$ and $B = U e_{\propto} S \; U \; f_{\beta} S$ where $e'_{\alpha} S$ and $f'_{\beta} S$ are idempotents in A and B respectively. On the other hand if S is a principal ideal semigroup then A and B are finitely generated so both ideals A and B are of the form aS and bS respectively for some a and b. Now a = bs for some s. So that aS = (bS) (sS). Hence S is a multiplication semigroup.

Counter example: The following example shows that the converse of this theorem need not be true.

Let $S = \{1, x, x^2, \ldots\ldots, e, f, ef\}$ with ex = x = fx = efx, $e=e^2$, $f=f^2$. It can be easily seen that S is a multiplication semigroup which is neither regular nor a principal ideal semigroup.

Remark : We now examine the converse. We shall determine the conditions under which multiplication semigroups are regular or principal ideal semigroups. At present we are not able to find the conditions for regularity in the case of arbitrary multiplication semigroups. However, we provide conditions for multiplication semigroups in which every ideal is a product of prime ideals to be regular. This class of semigroups includes all noetherian semigroups and some non-noetherian semigroups, the example for the later being $\{1, x_1, x_2, \ldots\ldots\}$ where $x_i x_j = x_{\min(i,j)}$.

1.2.5. Theorem : Let S be a multiplication semigroup in which every ideal is a product of primary ideals. Then S is regular iff every prime ideal is idempotent.

Proof: Let A be a prime ideal in a regular semigroup S. We have to show that A is idempotent i.e. $A^2 = A$. Clearly $A^2 \subseteq A$. For $a \in A$, we have a = axa, so $a \in A^2$ also. Thus $A \subseteq A^2$. Thus $A^2 = A$. Conversely, if A is any ideal then by hypothesis, $A = \pi_{i=l}^{n} Q_i$, where Q'_i s are primary ideals, then by (iv) of proposition 1.2.1, every $Q_i = P_i^{ni}$ where $P_i = \sqrt{Q}$ is a prime ideal for every i . Since every prime ideal P_i is an idempotent ideal, we must have then $A = \pi_{i=l}^{n} Pi = A^2$. Hence S is a regular semigroup.

1.2.6. Corollary : A noetherian multiplication semigroup S is regular iff every prime ideal is idempotent.

Proof: In view of theorem 1.2.5, it suffices to show that every ideal is a product of primary ideals in a noetherian multiplication semigroup S. As in commutative rings, in a noetherian semigroup it is easy to check that every ideal is a finite intersection of primary ideals [7,40]. Now, for any ideal A, $A = \cap_{i=1}^{n} Q_i$ where each Q_i is primary ideal. Now by (iv) of proposition 1.2.1, $Q_i = p_i^{ni}$ for some positive integer n_i, where $p_i = \sqrt{Q_i}$ is a prime ideal for every i. We can write $A = \cap_{i=1}^{n} P_i^{ni}$ with $P_i \not\subseteq P_j$ for i ≠ j. Since $\cap_{i=1}^{n} P_i^{ni} \subseteq P_i^{ni}$ for every i, by multiplication property, we have $\cap_{i=1}^{n} P_i^{ni} = P_1^{n1} A_1 \subseteq P_2^{n2}$ where A_l is some ideal of S. By (ii) of proposition 1.2.1, P_2^{n2} is P_2–primary so that $A_l \subseteq P_2^{n2}$ since $P_l^{nl} \not\subseteq P_2$ [see definition of primary ideal]. Then $A_1 = P_2^{n_2} A_2$ for some ideal A_2 . So by induction $\cap_{i=1}^{n} P_i^{ni} = \pi_{i=1}^{n} P_i^{ni} B$, where B is some ideal of S. Hence $A = \cap_{i=1}^{n} p_i^{ni} = \pi_{i=l}^{n} p_i^{ni}$. Thus every ideal is a product of primary ideals in a noetherian multiplication semigroup S. Hence by theorem 1.2.5, S is regular iff every prime ideal is idempotent.

1.2.7. Theorem : Let S be a multiplication semigroup satisfying any one of the following conditions :

(a) S Contains no idempotent prime ideals.

(b) Every idempotent prime ideal of S contains at most a finite number of prime ideals.

(c) No idempotent prime ideal is equal to the union of all the prime ideals properly contained in it.

(d) Every idempotent prime ideal of S is finitely generated.

Then S is a principal ideal semigroup iff prime ideals of S are linearly ordered.

Proof: Let S be a principal ideal semigroup, then it can be easily verified that prime ideals of S are linearly ordered.

Conversely, if prime ideals of S are linearly ordered, then every ideal has only one minimal prime divisor, so that the radical of every ideal is a prime ideal. Hence by (ii) and (iv) of proposition 1.2.1, every ideal is a power of a prime ideal. So it is sufficient to prove that every prime ideal is principal in order to show that S is a principal ideal semigroup. Since prime ideals of S are linearly ordered, by (x) of proposition 1.2.1 every non-idempotent prime ideal P is principal. So in fact we need only to prove that every idempotent prime ideal is principal.

Now if S satisfies (a), then clearly S is a principal ideal semigroup.

Let S satisfies (b). Let P be an idempotent prime ideal. If P does not contain any prime ideals then for any $x \in P$, P is the only minimal prime divisor of xS. Then by (ii) and (iv) of proposition 1.2.1, P= xS. Thus every idempotent prime ideal P is principal ideal. Hence S is a principal ideal semigroup. Now, suppose P contains prime ideals. Let $Q_1, Q_2, \ldots, Q_n$ be the set of all prime ideals with each $Q_i < P$. Then $\bigcup_{i=1}^{n} Q_i < P$. Now, for every $x \in P \setminus \bigcup_{i=1}^{n} Q_i$, it can be seen as before that P = xS.

Let S satisfy (c). If an idempotent prime ideal P does not contain any prime ideals then P is principal as in the preceding paragraph. On the other hand, if P contains prime ideals and if $\{Q_\alpha\}_\alpha$ is the set of all prime ideals with each $Q_\alpha < P$. Then $\bigcup_\alpha Q_\alpha < P$ and hence P = xS for any $x \in P \setminus \bigcup_\alpha Q_\alpha$ as in the preceding paragraph.

Finally, we show that (d) implies (c), which completes the proof. If P is a finitely generated idempotent prime ideal then we can write $P = \bigcup_{i=1}^{n} e_i S$.

Let $\{Q_\alpha\}_\alpha$ be the set of all prime ideals with each $Q_\alpha < P$. If $\bigcup_\alpha Q_\alpha = P$, then $e_i \in Q_{\alpha i}$ for some i , so that $P = \bigcup_{i=1}^{n} e_i S \subseteq \bigcup_{i=1}^{n} Q\alpha i < P$, which is a contradiction. Therefore $P \neq \bigcup_\alpha Q_\alpha$. Hence the proof.

Counter example: The following example shows that a multiplication semigroup S is not necessarily a principal ideal semigroup if it does not satisfy any one of the conditions (a)-(d) stated in the above theorem, even though its prime ideals are linearly ordered.

Set $S = \{1, x_1, x_2, \ldots\}$ with $x_i x_j = x_i$ for $i \leq j$. Then the ideal $\bigcup_{i=1}^{\infty} x_i S$ is not a principal ideal.

1.2.8. Theorem: For a finite-dimensional multiplication semigroup (S,M), the following are equivalent.

(i) S is a principal ideal semigroup.

(ii) Every idempotent prime ideal of S is principal.

(iii) Prime ideals of S are linearly ordered.

Proof: (i) ⇒ (ii) is evident.

(ii) ⇒ (iii) Let dim S = n. Then there exists a chain of prime ideals $P_0 < P_1 < \ldots < P_n = M$ with no prime ideals between P_i and P_{i-1} for every $i \geq 1$. If P_i is a non-idempotent prime ideal, then by (ix) of proposition 1.2.1., P_i contains properly at most one prime ideal Q with no prime ideals between P_i and Q. So that P_{i-1} is the only prime ideal properly contained in P_i with no prime ideals between P_i and P_{i-1} for $i \geq 1$. Now if P_j is an idempotent prime ideal, then by hypothesis P_j is principal ideal i.e. $P_j = eS$ for some idempotent element e. Suppose Q' is a prime ideal such that $Q' \neq P_{j-1}$, $Q' < P_j$, with no prime ideals between P_j and Q'. Then $Q' \cup P_{j-1} = P_j$ so that $e \in Q'$ or $e \in P_{j-1}$, i.e. $P_j \subseteq Q'$ or $P_j \subseteq P_{j-1}$ which is not true. Thus every idempotent prime ideal P_j also contains properly at most one prime ideal Q with no prime ideals between P_j and Q. So that P_{j-1} is the only prime ideal properly contained in P_j with no prime ideals between P_j and P_{j-1} for $j > 1$. It now follows that there are no prime ideals in S other than the P'_i s of the above chain. Thus prime ideals of S are linearly ordered.

(iii) ⇒ (i) : Since dim S = n, clearly every idempotent prime ideal of S contains properly at most a finite no. of prime ideals, and by hypothesis, prime ideals of S are linearly ordered, hence by theorem 1.2.7, S is a principal ideal semigroup.

1.2.9. Theorem: If (S,M) is a finite – dimensional semigroup, then S is a multiplication semigroup iff S is any one of the following types.

(i) S is a Dedekind semigroup containing only cancellable elements.

(ii) S is a principal ideal semigroup.

(iii) There exists an idempotent prime ideal P which is not principal, which is a multiplication subsemigroup of S (P may or may not contain identity) such that every ideal of S not contained in P is principal.

Proof: If S contains only cancellable elements, then from Corollary 1.2.3, it follows that S is of type (i). Suppose S contains non-cancellable elements also. If every idempotent prime ideal is principal, then S is of type (ii) by theorem 1.2.8. On the other hand, suppose there is an idempotent prime ideal which is not principal. If $\{P_\alpha\}$ is the set of all idempotent prime ideals which are not principal, then it is easy to verify that $\bigcup_\alpha P_\alpha$ is an idempotent prime ideal, say P, which is not principal. By (vi) of proposition 1.2.1, P is a multiplication subsemigroup of S, not necessarily containing identity. Now, we claim that the prime ideals not contained in P are linearly ordered. Since S is finite-dimensional we can have a chain of prime ideals P $=Q_O < Q_1 < \ldots < Q_t = M$ with no prime ideals between any two consecutive prime ideals. If $Q_i \neq Q_i^2$ then by (ix) of proposition 1.2.1, Q_{i-1} is the only prime ideal contained in Q_i with no prime ideals between Q_i and Q_{i-1}. If Q_j is an idempotent prime ideal, since Q_j is a principal ideal (as the prime ideals of S are linearly ordered, by (ii) of theorem 1.2.8, every idempotent prime ideal of S is principal) it follows from the proof of theorem 1.2.8 that Q_{j-1} is the only prime ideal properly contained in Q_j with no prime ideals between Q_j and Q_{j-1}. Thus, it is clear that Q_1, Q_2 ,... Q_t = M are the only prime ideals not contained in P, which are linearly ordered. Now using a similar proof as in (x) of proposition 1.2.1, we conclude that every non-idempotent prime ideal not contained in P is principal. Also by choice of P every idempotent prime ideal not contained in P is principal. Clearly every ideal not contained in P has only one minimal prime divisor, which is principal. Hence in view of (ii) and (iv) of proposition 1.2.1, every ideal not contained in P is principal.

Conversely, we have to prove that if S is any one of the types (i), (iii), (iii) then S is a multiplication semigroup.

A semigruop of type (i) is a multiplication semigroup by a well-known result.A semigroup of type (ii) is a multiplication semigruop by theorem 1.2.4.

Now suppose S is of type (iii). Let A and B be any two ideals of S with $A \subseteq B$. We have the following three possibilities : (i) A,B, $\subseteq$ P; (ii) $A \subseteq P$, $B \not\subseteq P$; (iii) A, B, $\not\subseteq$ P. In the first case A = BC for some ideal C of P, since A and B are trivially ideals of P with $A \subseteq B$. Now we can write A=B (SC) so that SC is an ideal of S. In the second case B = yS for some y; so that it is easy to verify that A=B (A:B) where A:B is clearly an ideal of S. Finally in the third case since A= aS $\subseteq$ B =bS we have a=bc for some c so that aS = (bS) (cS). Thus S is a multiplication semigroup in all three cases.

Remark: In proposition 1.2.1, We have seen that the conditions (ii) and (iv) are necessary for any multiplication semigroup S, while (viii) is seen to be a necessary condition when dim S$\leq$ 1. Even with this restriction on dimension of S, the above three conditions are not sufficient. However, with some additional hypothesis the sufficiency is established in the next theorem.

1.2.10. Theorem : Let dim S $\leq$ 1 and every non-maximal prime ideal be idempotent. Then (S,M) is a multiplication semigroup iff s satisfies the following conditions:

(i) Every ideal with prime radical is primary.
(ii) Every primary ideal is a power of its radical.
(iii) For every ideal A, A = ker A.

Proof: Let dim S $\leq$ 1 and (S,M) be a multiplication semigroup then by (ii), (iv) and (viii) of proportion 1.2.1, S satisfies the conditions (i), (ii) and (iii) respectively. Conversely, now suppose that S satisfies conditions (i), (ii) and (iii) and we have to prove that (S,M) is a multiplication semigroup. Let A and B be any two ideals with $A \subseteq B$. Since A = AS for any ideal A, it suffices to consider the case when A < B, with B $\neq$ S. If M is the only minimal prime divisor of both A and B then by (i) and (ii) A=M^k and B = M^ℓ for some positive integers k and ℓ where k > ℓ Setting C = $M^{k-\ell}$ we have A = BC . Now if $\{P_\propto\}$ is the set of all minimal prime divisors of A where each $P_\propto < M$, then it is easy to verify that $\{P_\propto\}$ is the set of all minimal prime divisors of AB too. By (ii), for each $\propto$, $P_\propto$ is the isolated $P_\propto$ - primary component of A and AB, since every non-maximal prime ideal is idempotent (by hypothesis). In view of (iii) A = $\cap P_\propto$= AB (see defn. of ker A). Hence S is a multiplication semigroup.

Remark: The following theorem shows that the conditions (i) and (ii) of theorem 1.2.10 are enough to determine a subclass of one-dimensional multiplication semigroups.

1.2.11 Theorem: - If S contains only two prime ideals which are different from S, then (S,M) is a multiplication semigroup iff S satisfies the following conditions;

(i) Every ideal with prime radical is primary.
(ii) Every primary ideal is a power of its radical.

Proof: Let (S, M) be a multiplication semigroup, then S satisfies (i) and (ii) by (ii) and (iv) of proposition 1.2.1.Conversely, suppose S satisfies the conditions (i) and (ii). By hypothesis, S contains only two prime ideals which are different from S, let them be P and M, with P < M. We have to show that S is a multiplication semigroup. As before it suffices to consider any pair of ideals A and B such that A < B with B $\neq$ S. If A and B have the same minimal prime divisors P or M, then in view of (i) and (ii) the multiplication property for the pair of ideals A and B can be easily verified. Suppose P and M are the minimal prime divisors of A and B respectively. Then by (i) and (ii) we can write A = P^k and B = M^ℓ for some positive integers k and ℓ. Since P is the only minimal prime divisor of the ideal $P^K.M^\ell$, by (i) and (ii) we get, $P^K.M^\ell = P^n$ for some positive integer n. By (i) P^n is P-primary. So that we have $P^k \subseteq P^n$ since $M^\ell \not\subseteq P$. Thus $P^k = P^n$ and hence $P^k . M^\ell = P^k$, i.e. AB = A. Hence S is a multiplication semigroup.

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Appendix

NOTATIONS AND SYMBOLS:

$x = y$	: x is equal to y
$a \neq b$	: a is not equal to b
$\|$	: such that
==>	: implies
$a \in S$	: a belongs to s
$x \notin P$	: x does not belong to P
$P < Q$	: P is properly contained in Q
$A \subseteq B$	: A is properly contained in or equal to B
$A \setminus B$	: the set of elements of A which are not in B
aS	: A set $\{ ax \mid x \in S \}$
A : B	: $\{ x \in S \mid xB \subseteq A \}$
$\cup P_i$	: union of sets P_i
$\cap P_i$	: Intersection of sets P_i
(S, M)	: Commutative semigroup S containing identity and having a unique maximal ideal M
$\sqrt{A}$	: Radical of A
A^w	: $\bigcap_{i=1}^{\infty} A^i$, for any ideal A
S_p	: $\{ a / b \mid a \in S, b \in S \setminus P$ is cancellable element $\}$
η	: mapping from $S \rightarrow S_p$ defined by $\eta(s) = s / 1$ ($s \in S$, where 1 is identity element of S)
AS_p	: The ideal generated by $\eta(A)$ is Sp, where A is any ideal of S.
Z	: Set of all non-cancellable elements.
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