A REVIEW ON QUEUEING SYSTEM WITH MULTI SERVER MODEL

P.JAYACHITRA, DR.A.JAMES ALBERT

Abstract : This paper deals with a review on multi server M/M/C type queue with a single vacation policy for some idle servers. In this queueing system if at a service completion instant, any d ($d \le c$) servers become idle, these d servers will take one and only one vacation together. During the vacation of d servers, the other (c-d) servers do not take vacation even if they are idle. Using the difference operator technique method we obtain the stationary distribution of the system. Conditional stochastic decomposition have been established for the waiting time and queue length given that all servers are busy.

Keywords : Multi server queue, conditional stochastic decomposition, single vacation policy, operator technique method.

Introduction: Several surveys on a single vacation models have been done by Doshi[1,2] . A Single vacation policy is more appropriate than a multiple vacation policy in the situation where the secondary job is performed once by these idle servers. For example a preventive maintenance action (a single vacation) can be performed on a block of machines (partial servers) when these machines are finished with processing jobs and become idle while other machines keep the system running. Tian and Cao[6] , Fuhrmann and Cooper[3] have presented a more detailed analysis of M/M/C vacation system and proved several conditional stochastic decomposition results for the queue length and customer waiting time. Zhang and Tian[4] have studied the vacation model with a partial server multiple vacation policy in which some servers (not all) take multiple vacations. They have also developed some explicit formulas for the stationary distributions without using iterative algorithm and proved some conditional stochastic decomposition properties.

The primary objectives of this paper are

To present a difference operator technique method for developing the stationary probabilities that the system contains n ≥ c customers while the server is busy or on vacation.

To calculate the waiting time and queue length give that all servers are busy using conditional stochastic decomposition.

Model Description: Consider an M/M/C queue with Poisson arrival rate λ and exponential services rate μ for each of all servers. in this system, any d (1 ≤ d < c) of the c servers may take vacation. The vacation policy prescribed that at a service completion instant, if any d servers become idle, these d servers will take a vacation together. During this vacation, the (c-d) server do not take any vacation even they are idle. After the vacation is completed, these d servers will either stay idle or serve the queue according to one of the following three cases.

1. There are no waiting customers: the d server stay idle and are ready for serving any new arriving customers;
2. There are j customers where c-d<j< c : j-c+d returning servers start serving customers immediately and c-j returning servers become idle;
3. There are j ≥ c customers in the system: the d servers all start serving the customers immediately.

The vacation time is assumed to be exponentially distributed with mean of 1/θ. The service order is assumed to be first come first served. In addition, inter-arrival times, services times, and vacation times are mutually independent.

State Space Of The Queuing System

Let P_{ij} (t) denote the probability that the system is in state(i,j) at time t where i denote the number of customers in the system at a time t and j denotes the status of the servers. If j=0, there are d servers on vacation at time t and if j=1, there are no servers on vacation at time t.

The steady state equations in the system size probabilities are given by:

$$-(\lambda+\theta)P_{00}+\mu P_{10}=0 \quad (1)$$

$$\lambda P_{n-1,0}-(\lambda+n\mu+\theta)P_{n0}+(n+1)\mu P_{n+1,0}=0$$
$$(1\le n<c-d) \quad (2)$$

$$\lambda P_{c-d-1,0}-(\lambda+\theta+(c-d)\mu)P_{c-d,0}+(c-d)\mu P_{c-d+1,0}+(c-d+1)\mu P_{c-d+1,1}=0$$
$$(3)$$

$$\lambda P_{n-1,0}-(\lambda+\theta+(c-d)\mu)P_{n0}+(c-d)\mu P_{n+1,0}=0$$
$$(c-d+1\le n\le c-1) \quad (4)$$

$$\lambda P_{n-1,0}-(\lambda+\theta+(c-d)\mu)P_{n0}+(c-d)\mu P_{n+1,0}=0$$
$$(n\ge c) \quad (5)$$

$$\theta P_{00}-\lambda P_{01}+\mu P_{11}=0 \quad (6)$$

$$\lambda P_{n-1,1}+\theta P_{n0}-(\lambda+n\mu)P_{n1}+(n+1)\mu P_{n+1,1}=0$$

$$(1 \le n < c-d) \tag{7}$$

$$\lambda P_{c-d-1,1} + \theta P_{c-d,0} - \left(\lambda + (c-d)\mu\right) P_{c-d,1} = 0 \tag{8}$$

$$\lambda P_{n-1,1} - \theta P_{n0} - \left(\lambda + n\mu\right) P_{n1} + \left(n+1\right)\mu P_{n+1,1} = 0$$
$$(c-d+1 \le n \le c-1) \tag{9}$$

$$\lambda P_{n-1,1} - \left(\lambda + c\mu\right) P_{n1} + c\mu P_{n+1,1} + \theta P_{n0} = 0$$
$$(n \ge c) \tag{10}$$

Steady state solution :

In the original paper Zhe Zhang and Naishuo Tian[5] have calculated the probabilities P_{n0} and P_{n1} for $n \ge c$ using matrix geometric method. Here we use the difference operator technique and get the expression for the probabilities P_{n0} and P_{n1} for $n \ge c$.

Introducing the forward shifting operator E defined by $E(P_{n0}) = P_{n+1,0}$ the equation (5) can be rewritten as

$$\left\{\left(c-d\right)\mu E^2 - \left[\lambda + \theta + (c-d)\mu\right] E + \lambda\right\} P_{n0} = 0$$
for $n \ge c-1$

The characteristics equation of the above differential equation is given by

$$\left(c-d\right)\mu z^2 - \left[\lambda + \theta + (c-d)\mu\right] z + \lambda = 0$$

It is easy to verify that the quadratic equation has two differential real roots r<r* with r<1 and r*>1. Since $\sum_{n=0}^{\infty} P_{n0} < 1$ the solution of the homogeneous difference equation is given by

$$P_{n0} = Ar^n \quad \text{for} \quad n \ge c-1$$

At n = c-1, $P_{c-1,0} = Ar^{c-1}$ implies $P_{n0} = r^{n-c+1} P_{c-1,0}$ for every $n \ge c-1$ (11) Equation (5) at n = c and Substituting for $P_{c+1,0}$ we have

$$\left[\lambda + \left(c-d\right)\mu r^2\right] P_{c-1,0} = \left[\lambda + \left(c-d\right)\mu + \theta\right] P_{c0}$$

Thus the equation (11) can be written as

$$P_{n0} = r^{n-c} P_{c0} \quad \forall n \ge c-1 \tag{12}$$

Similarly introducing the forward shifting operator for P_{n1} equation10 can be written as

$$c\mu E^2 - (\lambda + c\mu)E + \lambda = -\theta P_{n+1,0} \quad n \ge c-1 \tag{13}$$

This gives a non homogeneous difference equation with $P_{n+1,0} = r^{n+1-c} P_{c0}$ and the characteristics equation

$c\mu z - (\lambda + c\mu)z + \lambda = 0$ has roots 1 and $\dfrac{\lambda}{c\mu}$

Assuming $\rho = \dfrac{\lambda}{c\mu} < 1$ the complete solution of the non-homogeneous difference equation (13) is given by

$$P_{n1} = \left[A\rho^n - \frac{\theta r^{n+1-c}}{c\mu r^2 - (\lambda + c\mu)r + \lambda} \right] P_{c0}$$

Let $B = \dfrac{\theta r^{1-c}}{c\mu r^2 - (\lambda + c\mu)r + \lambda}$

Hence $P_{n1} = \left[A\rho^n + Br^n\right] P_{c0}$ for $n \ge c-1$ (14)

Equation (10) at n=c implies

$$\lambda P_{c-1,1} - (\lambda + c\mu)P_{c1} + c\mu P_{c+1,1} + \theta P_{c0} = 0$$

using (14) in the above equation and substituting for $P_{c-1,1}$ we have

$$AP_{c0} = \rho^{-c} P_{c1} + \frac{\theta r \rho^{-c}}{c\mu(1-r)(p-r)} P_{c0}$$

Substituting for AP_{c0} in equation (14) we have

$$P_{n1} = \rho^{n-c} P_{c1} + \frac{\theta r}{c\mu(1-r)(\rho - r)} \left(\rho^{n-c} - r^{n-c}\right) P_{c0} \quad n \ge c \tag{15}$$

If we define,

$P_n = \left(P_{n0}, P_{n1}\right)$ for $n \ge c$ then the equation together with (12) implies

$$P_n = \left(P_{n0}, P_{n1}\right) = \left(P_{c0}, P_{c1}\right) R^{n-c}$$

Where R is a 2 x 2 matrix given by

$$R^{n-c} = \begin{bmatrix} r^{n-c} & \dfrac{\theta r(\rho^{n-c} - r^{n-c})}{c\mu(1-r)(\rho - r)} \\ 0 & \rho^{n-c} \end{bmatrix} \quad n \ge c$$

The following theorems have been derived by Zhe Zhang and Naishuo Tian[6] for calculating stationary probability distribution.

Theorem 1 If $\rho < 1$, $P_n = (P_{n0}, P_{n1})$, $n \ge 0$,then the stationary probability distribution of the queueing model is

$$P_{n0} = \begin{cases} P_{00} \Psi_n & 0 \le n \le c-d \\ P_{00} \Psi_{c-d} r^{n-c+d} & c-d+1 \le n \le c \end{cases}$$

$$P_{n1} = \begin{cases} P_{00} \dfrac{1}{n!} \left(\dfrac{\lambda}{\mu}\right)^n \dfrac{\theta}{\lambda} \displaystyle\sum_{j=n}^{c-d} j! \left(\dfrac{\mu}{\lambda}\right)^j \sum_{r=0}^{j} \psi_\gamma & 0 \le n \le c-d \\ P_{00} \dfrac{1}{n!} \left(\dfrac{\lambda}{\mu}\right)^n \left\{ \dfrac{r\theta}{\lambda(1-r)} \psi_{c-d} \displaystyle\sum_{j=0}^{n-c+d-1} (c-d+j)! \left(\dfrac{\mu}{\lambda}\right)^{c-d+j} r^j \right. & c-d+1 \le n \le c \\ \left. + (c-d)! \left(\dfrac{\mu}{\lambda}\right)^{c-d} \dfrac{\theta}{\lambda} \displaystyle\sum_{r=0}^{c-d} \psi_\gamma \right\} & \end{cases}$$

For n > c, $P_n = (P_{n0}, P_{n1})$

$$P_n = P_{00}\left(r^{n-c+d}\psi_{c-d}, \beta_{c1}\rho^{n-c} + \frac{\theta r^{d+1}}{c\mu(1-r)}\psi_{c-d}\left(\frac{\rho^{n-c} - r^{n-c}}{\rho - r}\right)\right)$$

Where

$$\psi_0 = 1 \qquad \psi_n = \frac{\theta}{n\mu}\sum_{r=0}^{n-1}\psi_\gamma + \frac{\lambda}{n\mu}\psi_{n-1}, \qquad 1 \le n \le c-d$$

and $\beta_{c1} = P_{00}^{-1}P_{c1}$

$$P_{c1} = \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c \left\{ \frac{r\theta}{\lambda(1-r)}\psi_{c-d}\sum_{j=0}^{d-1}(c-d+j)!\left(\frac{\mu}{\lambda}\right)^{c-d+j} r^j + (c-d)!\left(\frac{\mu}{\lambda}\right)^{c-d}\frac{\theta}{\lambda}\sum_{y=0}^{c-d}\psi_\gamma \right\}$$

and P_{00} can be determined using the normalizing condition.

Theorem: 2

If $\rho < 1$ conditional queue length $L_q^{(c)}$ can be decomposed into the sum of two independent random variables.

$$L_q^{(c)} \equiv L_0^{(c)} + L_d$$

Where $L_0^{(c)}$ is the conditional queue length of the classical M/M/C system without vacation and has a geometric distribution with parameter ρ and L_d is the additional queue length due to the vacation effect and has the p.g.f.

$$L_d(z) = \frac{1}{\sigma}\left\{\beta_{c1} + z\frac{\theta r^{d+1}}{c\mu(1-r)}\psi_{c-d}\frac{1}{1-zr}\right\}$$ Where

$$\sigma = \beta_{c1} + \frac{\theta r^{d+1}}{c\mu(1-r)^2}\psi_{c-d}$$

Theorem: 3

If $\rho < 1$, conditional waiting time when all servers are busy, denoted by $W^{(c)}$ can be decomposed into the sum of two independent random variables.

$$W^{(c)} = W_0^{(c)} + W_d$$

Where $W_0^{(c)}$ is conditional waiting time in a classical M/M/C queue without vacations when all servers are busy and W_d is the conditional delay due to the server vacations W_d has the Laplace Steiltgers Transform.

$$W_d(s) = \frac{1}{\sigma}\left\{\beta_{c1} + \frac{\theta r^{d+1}}{c\mu(1-r)^2}\psi_{c-d}\frac{c\mu(1-r)}{s + c\mu(1-r)}\right\}$$

Based on theorem 1, we can obtain the stationary probability distribution of the number of customers at any time instant for M/M/C queuing system with single synchronous vacation for partial serves. Let L_v denote the number of customers in the system at any time for the model.

Then

$P[L_v = n] = P_{n0} + P_{n1} \quad for \quad n \ge 0$ and the probability that d servers are on vacation and the probability that no servers are on vacation are

$$P(J=0) = \sum_{n=0}^{\infty} P_{n0} = P_{00}\left[\sum_{n=0}^{c-d}\psi_n + \psi_{c-d}\frac{r}{1-r}\right]$$ and

$$\text{P (J=1)} = 1 - P_{00}\left[\sum_{n=0}^{c-d}\psi_n + \psi_{c-d}\frac{r}{1-r}\right]$$ respectively.

The probability that all servers are busy can be computed as

$$P[L_v \ge c, J=1] = \frac{P_{00}}{1-\rho}\left[\beta_{c1} + \frac{\theta r^{d+1}}{c\mu(1-r)^2}\psi_{c-d}\right] \quad (16)$$

Let $L_0^{(c)}$ and $W_0^{(c)}$ be the conditional queue length and waiting time respectively for the classical M/M/C queue without vacation. Then it is well known that $L_0^{(c)}$ has a geometric distribution with parameter ρ and its probability generating function is $L_0^c(z) = \dfrac{1-\rho}{1-\rho z}$

Based on theorem (2) we get the expected conditional queue length given that all servers are busy is

$$E\left(L_q^{(c)}\right) = \frac{\rho}{1-\rho}\frac{1}{\sigma}\frac{\theta r^{d+1}}{c\mu(1-r)^3}\psi_{c-d}$$

From theorem (3) ,we can get the expected conditional waiting time given that all servers busy

$$E(W^{(c)}) = \frac{1}{c\mu(1-\rho)} + \frac{1}{\sigma}\frac{\theta r^{d+1}}{(c\mu(1-\rho))^3}\psi_{c-d}\frac{1}{c\mu(1-r)}$$

Conclusion : The difference operator technique method used in this paper works efficiently for deriving the expression for the steady state equations by the system size probabilities. This model might be used to analysis the machine maintenance system which consists of multiple machines processing the same type of jobs. Using block inspection during the idle time (BIDIT) policy we perform the preventive maintenance as long as a certain number d of machines becomes idle. The analysis of this paper will provide a useful performance evaluation tool for designing the machine maintenance policy.

References:

1. B.Doshi, queueing system with vacations-A survey, queueing systems 1 (1986) 29-66.
2. B.Doshi, Single server queues with vacations in: Stochastic Analysis of communication systems, ed., H.Takagi (North-Holland, Amsterdam,1990)pp.217-265 .
3. S.W.Fuhrmann and R.B.Cooper, Stochastic decomposition in the M/G/1 queue with generalized vacations, Oper.Res.33 (1985)1117-1129.
4. Zhang Z.G. and Tian T.V,(2003) Analysis on multiserver queue with synchronous vacations of partial servers, performances evaluations 52,269-282
5. Zhe.C.Zhang and Naishuo Tian (2003) Analysis of Queueing system with single vacation for some servers, Queueing system,45, 161-175.
6. Tian.N, Li and J.Cao(1990) Conditional stochastic decomposition in M/M/C queue with servers for some vacations, stochastic models 15/2 367-377.

***

Ph.D, Research Scholar, Karpagam University, jayaasasi_akshu@rediffmail.com
Dean, Karpagam University, asiralbert@gmail.com