DISCUSSION ON THE CONVERGENCE/DIVERGENCE OF SERIES AND SEQUENCES

ADENEGAN, KEHINDE EMMANUEL, BALOGUN, FOLORUNSO OJO

Abstract : This paper critically examined sequences (functions whose domain is a set of natural numbers, N and the range is a subset of real numbers, $\Re$) and series (the partial sum of a sequence), their convergence and divergence. The paper briefly considered some basic applicable theorems to sequences and series in terms of their convergence or otherwise. Majorly, the four basic tests; D'Alembert's ratio test, Leibnitz test, integral test and n^{th} root test were considered using different sets of series. Results explicitly show how the sequences and series converge or diverge respectively and how more than one test can be applicable to just a series.

Keywords : Convergence, Divergence, Sequence and Series.

Introduction : A function whose domain is a set of natural numbers N and the range is a subset of real numbers $\Re$ is called a sequence. Thus a function S is defined as a sequence if S: N→$\Re$. It is generally denoted by $\{S_n\}$; n = 1,2,3... One readily recalls, from elementary mathematics, that a sequence is a set of numbers arranged according to a prescribed rule and such rule can be used to obtain other members of the sequence. On the other hand, a series is a partial sum of a sequence.

In mathematical analysis, sequences and series are of great importance with a lot of consequences as far as their convergence or divergence is concerned. Ponnusammy (2012) and Balogun (2006) respectively had a lot to say about this in their books, foundations of mathematical analysis and mathematical methods. In fact, the basic theory of convergence of series was worked out by the French mathematician, Augustine Louis Cauchy in the 1820s as rightly contributed by Berggren and Singer (2007). The theory and application of infinite series are important in virtually every branch of pure and applied mathematics. This paper is concerned about discussing the convergence or otherwise of the sequence and series with less emphasis on their types but briefly mentioned in passage with some applicable theorems.

Sequence

As defined above, a sequence is a function whose domain is a set of natural numbers N and the range is a subset of real numbers, $\Re$. A sequence is established or defined only if a rule is given that determines the nth term for every positive integer n and this rule may be given by a formula for the nth term. In elementary mathematics, important types of sequence include the arithmetic sequence and geometric sequence. In advance terms in analysis, we have various types which include; increasing sequence, decreasing sequence, monotone sequence, constant sequence, bounded sequence, Cauchy sequence, uniformly continuous sequence, etc.

A sequence $\{S_n\}$ is said to be Cauchy if given $\varepsilon > 0$, there exists $n(\varepsilon)$ such that $|S_m - S_n| < \varepsilon$ for all $m > n(\varepsilon)$ and all $n > n(\varepsilon)$. This is equivalent to given $\varepsilon > 0$, there exists an integer $n(\varepsilon)$ such that $|S_{n+k} - S_n| < \varepsilon \, \forall \, n > n(\varepsilon)$ and all $k \in N$. We shall still make mention of this later.

2.2 convergence/divergence of a sequence

Let $\{S_n\}$ be a sequence. If for a given $S_0 \in R$ and $\varepsilon > 0$ there exists N∈N such that n ≥ N implies$|S_n - S_0| < \varepsilon$; then sequence $\{S_n\}$ is said to be convergent and converges to S_0. S_0 is called the limit of the sequence. Symbolically, we write

$$\lim_{n\to\infty} S_n = S_0 \qquad (2.2)$$

We exemplify this as we see that the sequence $S_n = \left\{\frac{1}{n}\right\}$ n = 1,2,... has a limit 0. (i.e. $\lim_{n\to\infty} S_n = S_0 = 0.$). By the definition of a convergence sequence; $\lim_{n\to\infty} S_n = 0.$ Hence for a given $\varepsilon > 0$; there exists N ∈ N such that for all n ≥ N implies that $\frac{1}{n} < \varepsilon$. If we choose N so that $\frac{1}{N} < \varepsilon$. Then $\frac{1}{n} <$ holds since $\frac{1}{n} < \frac{1}{N}$ if n ≥ N. Thus, $\lim_{n\to\infty} \frac{1}{n} = 0$.

Similarly, the limit of the sequence $a_n = \frac{n^2+1}{n(n-1)}$ as $n \to \infty$ and $n \to -\infty$ gives value 1.

Theorem 2.1: A sequence converges to a unique limit.

Remark 2.1: A sequence is said to be convergent if it has a limit and divergent if otherwise.

A sequence $\{U_n\}$ is said to diverge to ∞ if for a given K > 0 there exists a natural number N such that $U_n <$ K for all n ≥ N. Symbolically, we write $\lim_{n\to\infty} U_n = -\infty$.

3.1Series The expression $U_1 + U_2 + U_3, + \ldots + U_n + =$

$\sum_{n=1}^{\infty} U_n$ is called the series of the sequence $\{U_n\}$.

Types of Series include; Alternating Series, Harmonic Series (this is a special series of the form $\sum_{n=1}^{\infty}\frac{1}{n}$ and it is usually divergent as we shall see later), P-Series (this is another special series just like the harmonic series. It is of the form $\sum_{n=1}^{\infty}\frac{1}{n^p}$ and is convergent unlike the harmonic as shall be shown later), Power Series, Taylor's Series, Maclaurin Series, Fibonacci Series, geometric series, etc.

3.2Convergence/Divergence Of Series

If the sequence $S_1, S_2,\ldots,S_n$ of the partial sums of a series converges to S, we write S = $\sum_{i=1}^{\infty} U_i$

The limit to which a convergent series converges is called the sum of the series. If the sum of the infinite series exists, it is given by S_∞. Hence, we say that a series is convergent if $\lim_{n\to\infty} S_n$ = S

It is said to diverge if $\lim_{n\to\infty} S_n = \infty$ Or if $\lim_{n\to\infty} S_n = -\infty$.

Remark 3.1: A series that is neither convergent nor divergent is called an oscillatory series.

A power series converges if |x| < R and diverges if |x| > R where the constant R is the radius of convergence of the series. The interval –R < x < R is called the interval of convergence of the series.

We shall begin discussion by considering the geometric series $a + ar + ar^2 + \quad + ar^{n-1}$.

This type of series will either converge or diverge depending on the behavior of r.

Now, suppose r = 1, then $S_n = na$. For a > 0, $\lim_{n\to\infty} S_n = \infty$ and for a < 0, $\lim_{n\to\infty} S_n = -\infty$.

So for all a ≠ 0, the series diverges.

Again, if r < 1, $\lim_{n\to\infty} S_n = \lim_{n\to\infty}\frac{a(1-r^n)}{1-r}$ (By the sum of geometric progression)

$$=\lim_{n\to\infty}\left(\frac{a}{1-r}-\frac{ar^n}{1-r}\right)$$

$$=\frac{a}{1-r}-0$$ because $\lim_{n\to\infty} r^n = 0$ for all r < 1 and is convergent.

Hence, $\lim_{n\to\infty} r^n = 0$ and if r > 1, $\lim_{n\to\infty} S_n$ does not exist.

In the same vein, the series $\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)}$ is convergent and that it converges to ½. Also, the series $\sum_{k=0}^{\infty}\frac{k}{k+1}$ and $\sum_{x=0}^{\infty}\sin x$ both diverge. We can put on note that if $a_k \to 0$, then $\sum_{k=0}^{\infty} a_k$ converges but there are divergent series for which $a_k \to 0$.

Then, does it imply that by the definition of limit existence, every series will be suspected to converge? This could be false! The harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ which diverges, proves the contradiction. This can be seen thus;

Let $\sum_{n=1}^{\infty}\frac{1}{n}$ be defined by $\{x_n\} = 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}$, $n \in N$; then $\{x_n\}$ is not a Cauchy sequence because the series does not conform to the definition of Cauchy (see Balogun, 2006). To see this, let m > n, then $x_m - x_n = \frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{m}$. Now, each of the m-n terms on the right hand side is greater than or equal to $\frac{1}{m}$ and hence, $x_m - x_n > \frac{m-n}{m}$. By considering the special case m=2n, we have $x_m - x_n > \frac{1}{2}$ which shows that $\{x_n\}$ is not a Cauchy sequence and hence that x_n diverges.

Theorem 3.1: The convergence of the series $\sum_{n=1}^{\infty} U_n$ implies that $\lim_{n\to\infty} U_n = 0$.

3.3Tests For Convergence Of Series

Like the contrast we had with the harmonic series, by mere inspection, one readily observes it will converge. As such, in order not to make such analytical error, the need for the various tests for the convergence. We shall consider basically four tests; the integral test, the D'Alembert ratio test, the Leibnitz test, and the n^{th} root test.

The Integral Test

In the integral test each term a_n of the positive term of the series of a monotonic decreasing function $a_{n+1} < a_n$ is replaced by f(n). The integral variable 'n' is then replaced by a continuous variable x to have a function f(x) which is defined for all values of x.

In essence, this test is useful when the terms have a derivative configuration.

Theorem 3.2: Let f(x) be a non-increasing function such that $f(x) \geq 0$; $x \in [1,\infty]$. then f(1) + f(2) + ... + = $\sum_{n=1}^{\infty} f(n)$ will converge if the improper integral $\int_1^{\infty} f(x)dx$ converges and will diverge if the improper integral $\int_1^{\infty} f(x)dx$ diverges.

Hence, by the integral test that the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$ and diverges for $p \leq 1$

Now, let us see by the integral test whether the harmonic series diverges. The harmonic series is of the form $\sum_{n=1}^{\infty} \frac{1}{n}$. Now, put $f(x) = \frac{1}{x}$. Note $\int_1^{\infty} \frac{dx}{x}$ diverges. Hence, by the integral test, $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges.

D'Alemberts Ratio Test

Consider the series $U_1+U_2+U_3+...+U_n+... = \sum_{n=1}^{\infty} U_n$

The series is said to converge if $\lim_{n\to\infty} \left|\frac{U_{n+1}}{U_n}\right| < 1$ and diverges if $\lim_{n\to\infty} \left|\frac{U_{n+1}}{U_n}\right| > 1$.

Remark 3.3: The Ratio Test fails to test for the convergence or divergence of a series if $\lim_{n\to\infty} \left|\frac{U_{n+1}}{U_n}\right| = 1$.

This test is effective with factorials and combinations of powers and factorials. If the terms are rational functions, it is inconclusive and difficult to apply. From the foregoing, the D'Alemberts test fails for the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ while the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the same test.

Now, let us see with the Ratio Test that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^n}{n^2}$ diverges to 2.

Let $a_n = \frac{(-1)^{n-1} 2^n}{n^2}$; $a_{n+1} = \frac{(-1)^{n-1+1} 2^{n+1}}{(n+1)^2} = \frac{(-1)^n 2^{n+1}}{(n+1)^2}$ and $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{2n^2}{(n+1)^2}\right|$

So, $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{2n^2}{(n+1)^2}\right| = 2$.

The series diverges. In fact, if we use the Leibnitz test (to be discussed later) for this, obviously it will diverge. Hence, one should not make a mistake of saying since the limit equals 2, it implies convergence. No, the limit is not for the series directly.

Now, let us compare this series $\sum_1^n \frac{1}{k!}$ with the harmonic series $\sum_1^n \frac{1}{k}$ discussed earlier.

By the ratio test, we observe that $a_k = \frac{1}{k!}$ and $a_{k+1} = \frac{1}{k+1!}$. Hence,

$$\frac{a_{k+1}}{a_k} = \frac{1}{(k+1)!} \times \frac{k!}{1} = \frac{k!}{(k+1)!} = \frac{k!}{(k+1)k!} = \frac{1}{k+1}.$$

Thus, $\lim_{n\to\infty} \frac{1}{k+1} \to 0$.

Showing that $\sum \frac{1}{k!}$ converges to 0 as $k \to \infty$. One stands the chance of concluding that this will also diverge since it has a resemblance of the harmonic series which is known to diverge.

The n^{th} Root Test

This test is used if powers are involved. Suppose that for the series $\sum_{n=1}^{\infty} U_n$; $\lim_{n\to\infty} |U_n|^{1/n} = r$. The series converges if $r < 1$ and diverges if $r > 1$. i.e. $\sum_{n=1}^{\infty} U_n$ converges if $\lim_{n\to\infty} |U_n|^{1/n} < 1$ and diverges if $\lim_{n\to\infty} |U_n|^{1/n} > 1$.

Remark 3.4: The n^{th} Root Test fails to determine the convergence or the divergence of a series if $r = 1$.

Leibnitz's Test

Leibnitz's test sometimes referred to as Alternating Series test is used for determining the convergence or otherwise of an alternating series.

An alternating series converges if

$$\left.\begin{array}{l}\text{(i) } |a_{n+1}| \le |a_n| \\ \text{and (ii) } \lim_{n\to\infty} |a_n| = 0 \text{ for } n \ge 1\end{array}\right\} \quad (*)$$

Now, let us use the Leibnitz Test to test for the convergence/divergence of the series. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{1+2^n}$.

Note, $|a_n| = \frac{1}{1+2^n}$ and $|a_{n+1}| = \left|\frac{(-1)^{n-1+1}}{1+2^{n+1}}\right| = \frac{1}{1+2^{n+1}}$

.

For all n ≥ 1, $\frac{1}{1+2^n} > \frac{1}{1+2^{n+1}}$. This implies that $|a_{n+1}| \le |a_n|$ for all n ≥ 1

Therefore $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{1}{1+2^n} = 0$.

Thus the series is convergent.

Remark 3.5: A series may be absolutely convergent and conditionally convergent.

A convergent series $\sum_{n=1}^{\infty} a_n$ is said to converge absolutely if $\sum_{n=1}^{\infty} |a_n|$ converges. It converges conditionally if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges.

Theorem 3.4: Every absolutely convergent series is convergent.

Let us numerically demonstrate the last remark and theorem by considering $a_n = \frac{(-1)^{n-1}}{n}$ for n= 1,2,3,...

It follows that $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ (1)

and $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \left|\frac{(-1)^{n-1}}{n}\right| = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ (2)

Remark 3.5: The series (1) converges and (2) diverges by Leibnitz's test.

Conclusion :

It is interesting studying sequences and series with their convergence or divergence. From our discussion, we see that mere inspection of a series or sequence may not give us actual conclusion if such will converge or diverge directly. Also, if one has used a test to verify the convergence or otherwise of a series, one can further confirm using other relevant test(s).

References :

1. Balogun, F. O. (2006). Mathematical Methods. KOKOKA Brothers Ltd. Nigeria. ISBN 978-37948-9-2.
2. Berggren, L and Singer, J. (2007). Sequences and Series. Microsoft Student 2008 [DVD], Redmond, WA, Microsoft Corporation 2007.
3. http://www.springer.com/978-0-8176-8291-0.
4. Ponnusamy, S. (2012). Foundations of Mathematical Analysis. DOI 10.1007/978-0-8176-8292-7 2, ©Springer Science+Business Media, LLC 2012.
5. Stroud, K. A. (2001). Engineering Mathematics, 8th Ed. Anthony Rowe Ltd. Wiltshire.
6. Webber, J. B. (1982). Mathematical Analysis, 4th Ed. Harpir Institute, London.

***

[1,2]Department of Mathematics, Adeyemi College of Education, Ondo, Ondo State, Nigeria.
[1]Lecturer II, A.C.E. Ondo and Doctoral (Financial Mathematics) Research Student of the University of Ibadan, Ibadan Nigeria. [1]adenegankehinde@gmail.com,+2348036576466. [2]Chief Lecturer; bkokoka@yahoo.co.uk,+2348035636345.