FIXED POINTS RESULT ON THREE G-METRIC SPACES

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Abstract : In this paper, we prove a unique fixed point result for composite mappings defined in three different G-metric spaces not all of which need to be continuous. Our result is the improvement of various results already existing for two and three metric spaces.

Keywords : Complete G- metric space, Fixed point ,G-Cauchy sequence , Mapping ϕ .

Introduction : It is well known that the contractive type conditions play an important role in the study of fixed point theory. The first intrusting result on fixed point for contractive type mappings was established by Banach Caccioppoli in 1922. The Banach contraction principle has been generalized by many mathematicians Kannan [7], Chatterjea [1], Sehgal [9].
Further Hardy and Rogers [5], Ciric [3], Singh [10] define some new contraction conditions in metric space. Fisher and Rao [4] , Khan et.al [8] , Jain R.[6] proved some fixed point results in two and three metric spaces.
Mustafa in collaboration with Sims [10] introduced a new notation of generalized metric space called G-metric space in 2006. He proved many fixed point results for a self mapping in G- metric space under certain conditions. Many researchers W. Shatanawi [11], Z. Mustafa [12] -[13] proved fixed point theorems satisfying certain contractive conditions in G-metric space.
In the present work we prove a unique fixed point result for three composite mappings in three complete G- metric spaces under certain contractive condition.
Now, we give preliminaries and basic definitions which are used through-out the paper.

Definition 1.1: Let X be a non empty set, and let $G: X\times X\times X \to R^{+}$ be a function satisfying the following properties:

(G_1) $G(x,y,z)=0$ if $x=y=z$ (G_2) $0<G(x,x,y)$ for all $x,y\in X$,with $x\neq y$

(G_3) $G(x,x,y)\le G(x,y,z)$ for all $x,y,z\in X$, with $y\neq z$

(G_4) $G(x,y,z)=G(x,z,y)=G(y,z,x)$ (Symmetry in all three variables)

(G_5) $G(x,y,z)\le G(x,a,a)+G(a,y,z)$, for all $x,y,z,a\in X$ (rectangle inequality)

Then the function G is called a generalized metric space, or more specially a G- metric on X, and the pair (X, G) is called a G–metric space.

Definition 1.2 : Let (X,G) be a G - metric space and let $\{x_n\}$ be a sequence of points of X , a point $x\in X$ is said to be the limit of the sequence $\{x_n\}$, if $\lim_{n,m\to+\infty} G(x,x_n,x_m)=0$, and we say that the sequence $\{x_n\}$ is G - convergent to x or $\{x_n\}$ G - converges to x .
Thus, $x_n \to x$ in a G - metric space (X,G) if for any $\in>0$ there exists $k\in N$ such that $G(x,x_n,x_m)<\in$, for all $m,n\ge k$

Proposition 1.3 : Let (X,G) be a G - metric space. Then the following are equivalent:
i) $\{x_n\}$ is G - convergent to x
ii) $G(x_n,x_n,x)\to 0$ as $n\to+\infty$
iii) $G(x_n,x,x)\to 0$ as $n\to+\infty$
iv) $G(x_n,x_m,x)\to 0$ as $n,m\to+\infty$

Definition 1.4 : Let (X,G) be a G - metric space. A sequence $\{x_n\}$ is called a G - Cauchy sequence if for any $\in>0$ there exists $k\in N$ such that $G(x_n,x_m,x_l)<\in$ for all $m,n,l\ge k$, that is $G(x_n,x_m,x_l)\to 0$ as $n,m,l\to+\infty.$

Proposition 1.5 : Let (X,G) be a G - metric space .Then the following are equivalent:
i) The sequence $\{x_n\}$ is G - Cauchy;
ii) For any $\in>0$ there exists $k\in N$ such that $G(x_n,x_m,x_m)<\in$ for all $m,n\ge k$

Proposition 1.6 : A G - metric space (X,G) is called G -complete if every G -Cauchy sequence is G -convergent in (X,G).

Proposition 1.7 : Let (X,G) be a G-metric space. Then $f: X\to X$ is G-continuous at $x\in X$, if and only if it is G-sequentially continuous at x , that is , whenever $\{x_n\}$ is G-convergent to x , $\{f(x_n)\}$ is G-convergent to $f(x)$.

Definition 1.8 : Let $\phi: R^{+}\to R^{+}$ be a mapping such that

i) ϕ is increasing ii) $\phi(t) < t$, for all $t > 0$ and iii) $\lim_{n\to\infty}\phi^n(t) = 0$, where $\phi^n(t)$ denotes the composition of $\phi(t)$ with itself n-times.

2. Main Result :

$$G_1(fghx_1, fghx_2, fghx_3) \le \phi\left\{\max.\begin{bmatrix} G_1(x_1,x_2,x_3), \\ G_1(x_1, fghx_1, fghx_2), \\ G_1(x_2, fghx_2, fghx_3), \\ G_1(x_3, fghx_3, fghx_1), \\ G_2(hx_1, hx_2, hx_3), \\ G_3(ghx_1, ghx_2, ghx_3)\end{bmatrix}\right\} \text{--(2.1.1)}$$

$$G_2(hfg\,y_1, hfgy_2, hfgy_3) \le \phi\left\{\max.\begin{bmatrix} G_2(y_1,y_2,y_3), \\ G_2(y_1, hfgy_1, hfgy_2), \\ G_2(y_2, hfgy_2, hfgy_3), \\ G_2(y_3, hfgy_3, hfgy_1), \\ G_3(gy_1, gy_2, gy_3), \\ G_1(fgy_1, fgy_2, fgy_3)\end{bmatrix}\right\} \text{--(2.1.2)}$$

$$G_3(ghfz_1, ghfz_2, ghfz_3) \le \phi\left\{\max.\begin{bmatrix} G_3(z_1,z_2,z_3), \\ G_3(z_1, ghfz_1, ghfz_2), \\ G_3(z_2, ghfz_2, ghfz_3), \\ G_3(z_3, ghfz_3, ghfz_1), \\ G_1(fz_1, fz_2, fz_3), \\ G_2(hfz_1, hfz_2, hfz_3)\end{bmatrix}\right\} \text{---(2.1.3)}$$

for all $x_1,x_2,x_3 \in X$, $y_1,y_2,y_3 \in Y$ and $z_1,z_2,z_3 \in Z$ and ϕ is a mapping as given by **Definition1.8** : Then fgh has a unique fixed point u in X , hfg has a unique fixed in v in Y and ghf has a unique fixed w in Z.

Proof : Let x_0 be an arbitrary point in X. Define sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in X , Y and Z respectively as follows:

$y_n = hx_{n-1}$, $z_n = gy_n$, $x_n = fz_n$ for $n = 1,2,3,-----$

Applying inequality (2.1.1) , we have ,

Theorem 2.1: Let (X,G_1), (Y,G_2) and (Z,G_3) be three complete G-metric spaces. If h is a continuous mapping of X into Y , g is a continuous mapping Y into Z and f is a mapping of Z into X satisfying the following inequalities:

$$G_1(x_n, x_{n+1}, x_{n+2}) = G_1(fghx_{n-1}, fghx_n, fghx_{n+1})$$

$$\le \phi\left\{\max.\begin{bmatrix} G_1(x_{n-1}, x_n, x_{n+1}), \\ G_1(x_{n-1}, fghx_{n-1}, fghx_n), \\ G_1(x_n, fghx_n, fghx_{n+1}), \\ G_1(x_{n+1}, fghx_{n+1}, fghx_{n-1}), \\ G_2(hx_{n-1}, hx_n, hx_{n+1}), \\ G_3(ghx_{n-1}, ghx_n, ghx_{n+1})\end{bmatrix}\right\}$$

$$= \phi\left\{\max.\begin{bmatrix} G_1(x_{n-1}, x_n, x_{n+1}), G_1(x_{n-1}, x_n, x_{n+1}), \\ G_1(x_n, x_{n+1}, x_{n+2}), G_1(x_{n+1}, x_{n+2}, x_n), \\ G_2(y_n, y_{n+1}, y_{n+2}), G_3(z_n, z_{n+1}, z_{n+2})\end{bmatrix}\right\}$$

$$= \phi\left\{\max.\begin{bmatrix} G_1(x_{n-1}, x_n, x_{n+1}), \\ G_2(y_{n-1}, y_n, y_{n+1}), \\ G_3(z_{n-1}, z_n, z_{n+1})\end{bmatrix}\right\} \text{----(2.1.4)}$$

Applying inequality (2.1.2) , we can write

$$G_2(y_n, y_{n+1}, y_{n+2}) = G_2(hfgy_{n-1}, hfgy_n, hfgy_{n+1})$$

$$\le \phi\left\{\max.\begin{bmatrix} G_2(y_{n-1}, y_n, y_{n+1}), \\ G_2(y_{n-1}, hfgy_{n-1}, hfgy_n), \\ G_2(y_n, hfgy_n, hfgy_{n+1}), \\ G_2(y_{n+1}, hfgy_{n+1}, hfgy_{n-1}), \\ G_3\left(gy_{n-1}, gy_n, gy_{n+1}\right), \\ G_1(fgy_{n-1}, fgy_n, fgy_{n+1})\end{bmatrix}\right\}$$

$$= \phi\left\{\max.\begin{bmatrix} G_2(y_{n-1}, y_n, y_{n+1}), \\ G_2(y_{n-1}, y_n, y_{n+1}), \\ G_2(y_n, y_{n+1}, y_{n+2}), \\ G_2(y_{n+1}, y_{n+2}, y_n), \\ G_3(z_{n-1}, z_n, z_{n+1}), \\ G_1(x_{n-1}, x_n, x_{n+1})\end{bmatrix}\right\}$$

$$= \phi\left\{\max.\begin{bmatrix} G_1(x_{n-1}, x_n, x_{n+1}), \\ G_2(y_{n-1}, y_n, y_{n+1}), \\ G_3(z_{n-1}, z_n, z_{n+1})\end{bmatrix}\right\} \text{----(2.1.5)}$$

Also applying inequality (2.1.3) , we will have

$$G_3(z_n, z_{n+1}, z_{n+2}) = G_3(ghfz_{n-1}, ghfz_n, ghfz_{n+1})$$

$$\leq \phi\left\{\max.\begin{bmatrix} G_3(z_{n-1}, z_n, z_{n+1}), \\ G_3(z_{n-1}, ghfz_{n-1}, ghfz_n), \\ G_3(z_n, ghfz_n, ghfz_{n+1}), \\ G_3(z_{n+1}, ghfz_{n+1}, ghfz_{n-1}), \\ G_1(fz_{n-1}, fz_n, fz_{n+1}), \\ G_2(hfz_{n-1}, hfz_n, hfz_{n+1}) \end{bmatrix}\right\}$$

$$= \phi\left\{\max.\begin{bmatrix} G_3(z_{n-1}, z_n, z_{n+1}), \\ G_3(z_{n-1}, z_n, z_{n+1}), \\ G_3(z_n, z_{n+1}, z_{n+2}), \\ G_3(z_{n+1}, z_{n+2}, z_n), \\ G_1(x_{n-1}, x_n, x_{n+1}), \\ G_2(y_n, y_{n+1}, y_{n+2}) \end{bmatrix}\right\}$$

$$= \phi\left\{\max\begin{bmatrix} G_1(x_{n-1}, x_n, x_{n+1}), G_2(y_{n-1}, y_n, y_{n+1}), \\ G_3(z_{n-1}, z_n, z_{n+1}) \end{bmatrix}\right\} \quad (2.1.6)$$

Now , By induction on using inequalities (2.1.4) , (2.1.5) and (2.1.6) , we will have

$$G_1(x_n, x_{n+1}, x_{n+2}) \leq \phi^n\left\{\max.\begin{bmatrix} G_1(x_0, x_1, x_2), \\ G_2(y_0, y_1, y_2), \\ G_3(z_0, z_1, z_{21}) \end{bmatrix}\right\}$$

$$G_2(y_n, y_{n+1}, y_{n+2}) \leq \phi^n\left\{\max.\begin{bmatrix} G_1(x_0, x_1, x_2), \\ G_2(y_0, y_1, y_2), \\ G_3(z_0, z_1, z_2) \end{bmatrix}\right\}$$

$$G_3(z_n, z_{n+1}, z_{n+2}) \leq \phi^n\left\{\max.\begin{bmatrix} G_1(x_0, x_1, x_2), \\ G_2(y_0, y_1, y_2), \\ G_3(z_0, z_1, z_2) \end{bmatrix}\right\}$$

Since $\lim_{n\to\infty}\phi^n(t) = 0$, it follows that $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are G - Cauchy sequences with limits u , v and w in X , Y and Z respectively.

Since h and g are continuous , we have

$\lim_{n\to\infty} y_n = \lim_{n\to\infty} hx_{n-1} = hu = v$ and

$\lim_{n\to\infty} z_n = \lim_{n\to\infty} gy_n = w$

Again by using inequality we can write ,

$$G_1(fghu, fghu, x_n) = G_1(fghu, fghu, fghx_{n-1})$$

$$\leq \phi\left\{\max.\begin{bmatrix} G_1(u, u, x_{n-1}), G_1(u, fghu, fghu), \\ G_1(u, fghu, fghx_{n-1}), G_1(x_{n-1}, fghx_{n-1}, fghu), \\ G_2(hu, hu, hx_{n-1}), G_3(ghu, ghu, ghx_{n-1}) \end{bmatrix}\right\}$$

Since g and h are continuous , so on taking limit as $n \to \infty$ in the above inequality , we get

$G_1(fghu, fghu, u) \leq \phi(G_1(u, fghu, fghu))$, which implies that $fghu = u$, as $\phi(t) < t$.

Hence u is a fixed point of fgh .

We also have , $hfgv = hfghu = hu = v$, i.e. v is a fixed point of hfg

and $ghfw = ghfgv = gv = w$, i.e. w is a fixed point of ghf .

Now , we prove the uniqueness of the fixed point u .

Let us assume that α is another fixed point of fgh .

Therefore by using inequality (2.1.1) we can write ,

$$G_1(u, u, \alpha) = G_1(fghu, fghu, fgh\alpha)$$

$$\leq \phi\left\{\max.\begin{bmatrix} G_1(u, u, \alpha), \\ G_1(u, fghu, fghu), \\ G_1(u, fghu, fgh\alpha), \\ G_1(\alpha, fgh\alpha, fghu), \\ G_2(hu, hu, h\alpha), \\ G_3(ghu, ghu, gh\alpha) \end{bmatrix}\right\}$$

$$= \phi\left\{\max.\begin{bmatrix} G_2(hu, hu, h\alpha), \\ G_3(ghu, ghu, gh\alpha) \end{bmatrix}\right\} \quad \text{--} (2.1.7)$$

Again by using inequality (2.1.2) , we have

$$G_2(hu, hu, h\alpha) = G_2(hfghu, hfghu, hfgh\alpha)$$

$$\leq \phi\left\{\max.\begin{bmatrix} G_2(hu, hu, h\alpha), \\ G_2(hu, hfghu, hfghu), \\ G_2(hu, hfghu, hfgh\alpha), \\ G_2(h\alpha, hfgh\alpha, hfghu), \\ G_3(ghu, ghu, gh\alpha), \\ G_1(fghu, fghu, fgh\alpha) \end{bmatrix}\right\}$$

$$= \phi\left\{\max.\begin{bmatrix} G_1(u, u, \alpha), \\ G_3(ghu, ghu, gh\alpha) \end{bmatrix}\right\} \quad \text{--}(2.1.8)$$

Hence by using (2.1.7) and (2.1.8) , we have

$$G_1(u, u, \alpha) < \phi(G_3(ghu, ghu, gh\alpha))$$

Also finally using inequality (2.1.3) we can write

$$G_1(u, u, \alpha) \leq \phi(G_3(ghu, ghu, gh\alpha))$$

$$= \phi(G_3(ghfghu, ghfghu, ghfgh\alpha))$$

$$\leq \phi^2 \left\{ \max . \begin{bmatrix} G_3(ghu, ghu, gh\alpha), \\ G_3(ghu, ghfghu, ghfghu), \\ G_3(ghu, ghfghu, ghfgh\alpha), \\ G_3(gh\alpha, ghfgh\alpha, ghfghu), \\ G_1(fghu, fghu, fgh\alpha), \\ G_2(hfghu, hfghu, hfgh\alpha) \end{bmatrix} \right\}$$

$$=\phi^2 \left\{ \max . \begin{bmatrix} G_3(ghu, ghu, gh\alpha), G_1(u,u,\alpha), \\ G_2(hu, hu, h\alpha) \end{bmatrix} \right\}$$

$$=\phi^2 \left(G_1(u,u,\alpha)\right)$$

Since $\phi(t) < t$, it follows that $u = \alpha$ and hence the uniqueness of u is proved.

Similarly we can prove that v is the unique fixed point of hfg and w is the unique fixed point of ghf .

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