FIXED POINT RESULTS FOR CHATTERJEE TYPE CYCLIC WEAK CONTRACTIONS IN B-METRIC SPACES

K.MANJU

Abstract : The notion of Chatterjee type cyclic weak contraction is introduced and a fixed point theorem for Chatterjee type cyclic weak contraction is proved in a b-metric space . Our results extends and generalises many well known results.

Keywords : Fixed points, b-metric space, cyclic weak contraction

Introduction : Banach's contraction principle has played an important role in the development of non linear analysis, particularly the fixed point theory. Fixed point theorems has got wide applications not only in various branches of mathematics but also in Computer Science, Economics, Biology etc. Due to its importance many researchers in recent years have investigated the extension and generalization of Banach contraction principle in different directions. Generalizing the concept of metric space Bakhtin[3] introduced the concept of b-metric space which is not necessarily Hausdorff and proved the Banach contraction principle in the setting of a b-metric space. Since then, several papers have dealt with fixed point theory or the variational principle for single-valued and multi-valued operators in b-metric spaces (see [4,5,6,9,10,11,12] and the references therein). Note that spaces with non Hausdorff topology plays an important role in Tarskian approach to programming language semantics used in computer science (For some details see [18]). On the other hand in 2003 Kirk et al [16] gave an interesting generalization of Banach contraction principle by introducing the notion of cyclic representation of a metric space with respect to an operator $T : X \to X$. Since then many new fixed point theorems for for operators T defined on a complete metric space X with a cyclic representation of X with respect to T appeared in literature. (See [7,8,13,14,17,19,20]).

In this paper we have introduced the notion of Chatterjee type cyclic weak contraction and proved a fixed point theorem for Chatterjee type cyclic weak contraction in a b-metric space. Our results extends and generalises may well known results.

Preliminaries :

Definition 2.1 [3] : Let X be a nonempty set and the mapping $d : X \times X \to [0,\infty)$ satisfies:

(bM-1) $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$;

(bM-2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(bM-3) there exist a real number $s \geq 1$ such that $d(x, y) \leq s[d(x,z) + d(z, y)]$ for all $x, y, z \in X$.

Then d is called a b-metric on X and (X,d) is called a **b-metric space (in short bMS)** with coefficient s .

Note that every metric space is a b-metric space. However the converse is not necessarily true.

For any $x \in X$, the open ball with centre x and radius $r > 0$ is given by $B_r(x) = \{y \in X : d(x, y) < r\}$. The open balls in bMS are not necessarily open.

Let U be the collection of all subsets A of X satisfying the condition that for each $x \in A$ there exist $r > 0$ such that $B_r(x) \subseteq A$. Then U defines a topology for the bMS (X,d) , which is not necessarily Hausdorff.

Definition 2.2 : Let (X,d) be a b-metric space, $\{x_n\}$ be a sequence in X and $x \in X$. Then

(a) The sequence $\{x_n\}$ is said to be convergent in (X,d) and converges to x , if for every $\varepsilon > 0$ there exists $n_0 \in$ □ such that $d(x_n, x) < \varepsilon$ for all $n > n_0$ and this fact is represented by $\lim_{n\to\infty} x_n = x$ or $x_n \to x\$as\$n \to \infty$.

(b) The sequence $\{x_n\}$ is said to be Cauchy sequence in (X,d) if for every $\varepsilon > 0$ there exists $n_0 \in$ □ such that $d(x_n, x_{n+p}) < \varepsilon$ for all $n > n_0, p > 0$ or equivalently, if $\lim_{n\to\infty} d(x_n, x_{n+p}) = 0$ for all $p > 0$.

(c) (X,d) is said to be a complete b-metric space if every Cauchy sequence in X converges to some $x \in X$.

Main Results : Let F denote set of all lower semi continuous functions $\phi : [0,\infty)^2 \to [0,\infty)$ satisfying $\phi(x, y) = 0$ if and only if $x = y = 0$ and Θ denote the class of continuous functions $\theta : [0,\infty)^4 \to [0,\infty)$ satisfying $\theta(t_1,t_2,t_3,t_4) = 0$ whenever a) $t_1 = t_3 = 0$ OR b) $t_2.t_4 = 0$

Example 3.1 The following functions belong to the class Θ. For $t_1,t_2,t_3,t_4 \geq 0$

(1) $\theta(t_1,t_2,t_3,t_4) = \min\{t_1+t_3, t_2, t_4\}$, .

(2) $\theta(t_1,t_2,t_3,t_4) = \ln(1+(t_1+t_3)t_2t_4)$.

(3) $\theta(t_1,t_2,t_3,t_4) = \ln(1+t_1+t_3)$ $\ln(1+t_2)\ \ln(1+t_4)$.

(4) $\theta(t_1,t_2,t_3,t_4) = (t_1+t_3)t_2t_4$.

(5) $\theta(t_1,t_2,t_3,t_4) = e^{(t_1+t_3)t_2t_4} - 1$.

Definition 3.2 Let (X,d) be a complete db-metric space, $A_1, A_2, \cdots A_m$ be non empty closed subsets of X and $Y = \bigcup_{i=1}^{m} A_i$. An operator $T: Y \to Y$ is called a Chatterjee type weak cyclic contraction if

(1) $\bigcup_{i=1}^{m} A_i$ is a cyclic representation of Y with respect to T, i.e. $T(A_i) \subseteq T(A_{i+1})$, $i = 1,2...m$ and $A_{m+1} = A_1$.

(2) For any $x \in A_i, y \in A_{i+1}$, $i = 1,2,\cdots m$, $\phi \in F$, $L \geq 0$, $A_{m+1} = A_1$, $\alpha \in [0, \frac{1}{s})$

$$d(Tx,Ty) \leq \alpha(d(x,Ty)+d(y,Tx)) -\phi(d(x,Ty),d(y,Tx)) +L\theta(d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx))$$

Now we present our main results as follows :

Theorem 3.3 Let (X,d) be a b-metric space with coefficient $s \geq 1$, $A_1, A_2, \cdots A_m$ be non empty closed subsets of X and $Y = \bigcup_{i=1}^{m} A_i$. Suppose that $T: Y \to Y$ is a Chatterjee type weak cyclic contraction. Then T has a unique fixed point $z \in \bigcap_{i=1}^{m} A_i$.

Proof For some $x_0 \in Y$ by (1) we construct the sequence $\{x_n\}$ given by $x_{n+1} = Tx_n, n \geq 0$. If there exist $n_0 \in N$ such that $x_{n_0+1} = x_{n_0}$ then the proof is finished. Suppose $x_{n+1} \neq x_n$ for any $n = 0,1,2,\cdots$. Note that for $n \geq 0$ there exist $i_n \in \{1,2,\cdots m\}$ such that $x_n \in A_{i_n}$ and $x_{n+1} \in A_{i_n+1}$. Now since T is Chatterjee type weak cyclic contraction we have

$$d(x_n,x_{n+1}) = d(Tx_{n-1},Tx_n)$$

$$\leq \alpha(d(x_{n-1},Tx_n)+d(x_n,Tx_{n-1})) - \phi(d(x_{n-1},Tx_n),d(x_n,Tx_{n-1})) +L\theta(d(x_{n-1},Tx_{n-1}),d(x_n,Tx_n),d(x_{n-1},Tx_n),d(x_n,Tx_{n-1}))$$

$$\leq \alpha(d(x_{n-1},x_{n+1})+d(x_n,x_n)) - \phi(d(x_{n-1},x_{n+1}),d(x_n,x_n))$$

$$+L\theta(d(x_{n-1},x_n),d(x_n,x_{n+1}),d(x_{n-1},x_{n+1}),d(x_n,x_n))$$

$$\leq \alpha(s.[d(x_{n-1},x_n)+d(x_n,x_{n+1})]) - \phi(d(x_{n-1},x_{n+1}),0)$$

$$\leq s\alpha(d(x_{n-1},x_n)+d(x_n,x_{n+1})).$$

Thus we have

$$d(x_n,x_{n+1}) \leq \frac{s\alpha}{1-s\alpha} d(x_{n-1},x_n) \text{ for all } n = 1,2,\cdots$$
$$= kd(x_{n-1},x_n)$$

Where $k = \frac{s\alpha}{1-s\alpha} < 1$. Continuing this process we get $d(x_n,x_{n+1}) \leq k^n d(x_0,x_1)$. Hence for all $n \geq 1$ and $p > 0$ we have

$$d(x_n,x_{n+p}) \leq s[d(x_n,x_{n+1})+d(x_{n+1},x_{n+p})]$$
$$\leq sd(x_n,x_{n+1})+s^2d(x_{n+1},x_{n+2})+\cdots+s^p d(x_{n+p-1},x_{n+p})$$

$$\leq sk^n d(x_0,x_1)+s^2k^{n+1}d(x_0,x_1)+\cdots+s^p k^{n+p-1}d(x_0,x_1)$$

$$= sk^n(1+sk+\cdots(sk)^{p-1})d(x_0,x_1)$$
$$\leq sk^n \frac{1}{1-sk} d(x_0,x_1).$$

Thus $d(x_n,x_{n+p}) \to 0$ as $n \to \infty$, for all $p > 0$ and so sequence $\{x_n\}$ is a Cauchy sequence. Since Y is the finite union of closed sets is closed and hence (Y,d) is a complete subspace of (X,d) and as $\{x_n\}$ is a Cauchy sequence in the complete subspace (Y,d), converges to some point say x in Y. Now since $Y = \bigcup_{i=1}^{m} A_i$ is a cyclic representation of Y with respect to T, the sequence $\{x_n\}$ has infinite terms in each A_i for each $i \in \{1,2\cdots m\}$. On the other hand (Y,d) is complete and so in each A_i, $i = 1,2\cdots m$ we can construct subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converge to x. Since A_i $i = 1,2\cdots m$ are closed we see that $x \in \bigcap_{i=1}^{m} A_i$. Again taking $x_{n_k} \in A_i$ and $x \in A_{i+1}$ we have

$$d(x_{n_k}, Tx) = d(Tx_{n_k-1}, Tx)$$
$$\leq \alpha(d(x_{n_k-1}, Tx) + d(x, Tx_{n_k-1})) - \phi(d(x_{n_k-1}, Tx), d(x, Tx_{n_k-1}))$$
$$+L\theta(d(x_{n_k-1}, Tx_{n_k-1}), d(x, Tx), d(x_{n_k-1}, Tx), d(x, Tx_{n_k-1}))$$
$$\leq \alpha(d(x_{n_k-1}, Tx) + d(x, x_{n_k})) - \phi(d(x_{n_k-1}, Tx), d(x, x_{n_k}))$$
$$+L\theta(d(x_{n_k-1}, x_{n_k}), d(x, Tx), d(x_{n_k-1}, Tx), d(x, x_{n_k})).$$

Letting $n \to \infty$ and using lower semi continuity of ϕ and θ we get

$$d(x, Tx) = \alpha(d(x, Tx))$$
$$- \liminf_{n \to +\infty} \phi(d(x_{n_k-1}, Tx), d(x, x_{n_k}))$$
$$+L.\liminf_{n \to +\infty} \theta(d(x_{n_k-1}, x_{n_k}), d(x, Tx), d(x_{n_k-1}, Tx), d(x, x_{n_k}))$$
$$= \alpha(d(x, Tx)) - \phi(d(x, Tx), 0) + L\theta(0, d(x, Tx), d(x, Tx), 0)$$

$\leq \alpha(d(x, Tx))$ which is a contradiction unless $d(x, Tx) = 0$. Hence x is a fixed point of T. Now suppose x and y are two fixed points of T in $\bigcap_{i=1}^{m} A_i$. Then we have

$$d(x, y) = d(Tx, Ty)$$
$$\leq \alpha(d(x, Ty) + d(y, Tx)) - \phi(d(x, Ty), d(y, Tx))$$
$$+L\theta(d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx))$$
$$= \alpha(d(x, y) + d(y, x)) - \phi(d(x, y), d(y, x))$$
$$+L\theta(d(x, x), d(y, y), d(x, y), d(y, x))$$
$$= \alpha(d(x, y) + d(y, x)) - \phi(d(x, y), d(y, x)) < 2\alpha d(x, y)$$

.

which is a contradiction unless $\phi(d(x, y), d(y, x)) = 0$. Hence using the property of ϕ we see that $x = y$. Hence the fixed point is unique.

Corollary 3.4 Let (X, d) be a b-metric space with coefficient $s \geq 1$, $A_1, A_2, \cdots A_m$ be non empty closed subsets of X and $Y = \bigcup_{i=1}^{m} A_i$. Suppose that $T: Y \to Y$ is an operator such that

(1) $\bigcup_{i=1}^{m} A_i$ is a cyclic representation of Y with respect to T.

(2) For any $x \in A_i, y \in A_{i+1}$, $i = 1, 2, \cdots m$, $\phi \in F$, $L \geq 0$, $A_{m+1} = A_1$, $\gamma \in [0, \frac{1}{2s})$

$$d(Tx, Ty) \leq \gamma(d(x, Ty) + d(y, Tx))$$
$$+L\theta(d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx))$$

Then T has a unique fixed point $z \in \bigcap_{i=1}^{m} A_i$.

Proof Proof follows from Theorem 3.3 by taking $\phi(x, y) = (\alpha - \gamma)(x + y)$, with $\alpha \in [\gamma, \frac{1}{2s})$.

Example 3.5 Let $X = \{1, 2, 3\}$. Define $d: X \times X \to E$ by $d(x, y) = 0$ if $x = y$, $d(1,2) = d(2,1) = 10, d(1,3) = d(3,1) = \frac{30}{7}$ and $d(2,3) = d(3,2) = \frac{24}{7}$. Then it is an easy exercise to see that (X, d) is a b-metric space with coefficient s = 2 but not a metric space. Let $A = \{1, 2\}$ and $B = \{1, 3\}$. Consider the mapping $T: A \bigcup B \to A \bigcup B$ given by $T1 = T2 = 1$ and $T3 = 2$. Define $\theta(t_1, t_2, t_3, t_4) = \min\{t_1 + t_3, t_2, t_4\}$, $t_1, t_2, t_3, t_4 \geq 0$

.

Then T satisfies all the conditions of Corollary 3.4 and 1 is the unique fixed point of T. However T does not satisfy contractions of Karapinar and Nashine\cite{karap3} at x = 2 and y = 3 as

$$d(Tx, Ty) = d(1, 2) = 10 > \frac{30}{7} = d(x, Ty) + d(y, Tx)$$

.

Remark 3.6 Clearly Theorem 2.2 of Karapinar and Nashine[15] is a particular case of our Theorem 3.3 with s=1 and L=0.

Remark 3.7 Many fixed point results can be derived from Theorem 3.3 and Corollary 3.4 with respect to particular choice of θ as given in Example 3.1.

References:

1. Aydi H, Bota M.F, Karapinar E, Moradi S, A common fixed pointfor weak ϕ-contractions on b-metric spaces, Fixed Point Theory, 13(2), 337-346,2012.
2. Aydi H, Bota M.F, Karapinar E, Mitrovic S, A fixed point theorem for set valued quasi contraction in b-metric space, Fixed Point Theory Appl. 2012:88, 2012.
3. Bakhtin I.A., The contraction mapping principle in quasimetric spaces, Funct. Anal., Unianowsk Gos. Ped. Inst. 30 (1989), 26â€“37.
4. Boriceanu M, Strict fixed point theorems for multivalued operators in b-metric spaces, International J. Modern Math., 4(3), 285-301, 2009.
5. Boriceanu M., Bota M., Petrusel A., Mutivalued fractals in b-metric spaces, Cen. Eur. J. Math. 8(2)(2010), 367-377.
6. Bota M., Molnar A., Csaba V., On Ekeland's variational principle in b-metric spaces, Fixed Point Theory, 12(2011), 21-28.
7. Chen C.M, Fixed point theory for cyclic weaker Meir Keiler function in complete metric spaces, Fixed Point Theory and Applications,Article 17, 2012.
8. Choudhary B S, Unique fixed point theorems for weak C-contractive mappings, Kathmandu Univ. J of Sci. Engineering Tech., 5(1), 6-13, 2009.
9. Czerwik S, Contraction mappings in b-metric spaces, Acta. Math. Inform. Univ. Ostraviensis, 1, 5-11, 1993.
10. Czerwik S., Nonlinear set-valued contraction mappings in b-metric spaces, Atti Sem. Mat. Univ. Modena, 1998, 46, 263-276.
11. Czerwik S, Dlutek K, Singh S.L, Round-off stability of iteration procedures for operators in b-metric spaces, J. Natur. Phys. Sci. , 11, 87-94, 1997.
12. Czerwik S, Dlutek K, Singh S.L, Round-off stability of iteration procedures for set valued operators in b-metric spaces, J. Natur. Phys. Sci. , 15, 1-8, 2001.
13. Karapinar E, Fixed point theory for cyclic weak ϕ-contraction, Appl. Math. Lett., 24(6), 822-825,2011.
14. Karapinar E, Erhan I.M, Ulus A.Y, {\it Fixed point theorem for cyclic maps on partial metric space, Appl. Math. Info. Sci., 6(1), 239-244, 20112
15. Karapinar E, Nashine H.K, Fixed point theorems for cyclic Chatterjee type contractions, J. Appl. Math., Article ID:165698, 2012.
16. Kirk W.A, Srinivasan P.S, Veeramani P, Fixed points for mappings satisfying cyclical contractive conditions, Fixed Point Theory, 4(1), 79-89, 2003.
17. Kosuru G.S.R Veeramani P, Cyclic contractions and best proximity point theorems, http://128.84.158.119/abs/1012.1434v2, 2011.
18. Mathews S.G, Partial Metric Topology, Papers on general topology and applications, Eighth summer conference at Queens college, Annals of New York Academy of Sciences, Vol 728, 183-197.
19. Pacurar M Rus I.A, Fixed point theory for cyclic weak ϕ-contraction, Nonlinear Analysis TMA, 72(3-4), 2683-2693, 2010.
20. Rus I.A, Cyclic representations and fixed points, Annals. Tiber. Popovic. Seminar of Functional Equations, Approximation and Convexity, vol 3, 171-178, 2005.

*****

MCA Department, Kristu Jyothi College of Management and Technology
Chengnacherry, Kerala, India
Email : manjukaran17@rediffmail.com