RINGS WITH COMMUTATORS AND THE SQUARE OF EVERY ELEMENT IS IN THE NUCLEUS

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Abstract : In this paper, we prove that (x,y,z) + (y,z,x) + (z,x,y) is in the nucleus N for all elements x,y,z of the ring R. Using this, we prove that a prime ring of char.≠2 is either associative or a Lie ring.

Keywords : Nonassociative ring, Prime ring, Nucleus, Center, Characteristic and Associator.

Introduction : Kleinfeld [2] considered rings in which the square of every element is in the nucleus, a property that is shared by both associative and Lie rings. Under the additional assumptions of primeness and char. ≠2 it was shown that such rings are either associative or have the property that $x^2 = 0$, for every element x of the ring. In this paper we prove that (x,y,z) + (y,z,x) + (z,x,y) is in the nucleus for all elements x,y,z of the ring R. Using this, it is proved that if R is a prime ring of char. ≠2, then it is either associative or a Lie ring. At the end of this section we give an example in which char. ≠2 is necessary in theorem 1.

Priliminaries : Let R be a nonassociative ring. We shall denote the commutator and the associator by (x,y) = xy-yx and (x,y,z) = (xy)z-x(yz) for all x,y,z in R respectively. The nucleus N of a ring R is defined as N = {n ∈ R / (n,R,R) = (R,n,R) = (R,R,n) = 0} [1]. The center C of R is defined as C = { c ∈ N / (c, R) = 0}. A ring R is said to be of characteristic □ n if nx = 0 implies x = 0, for all x ∈ R and n is a natural number. A ring R is of characteristic □ n is simply denoted by char.□□□n. A ring R is said to be prime if whenever A and B are ideals of R such that AB = 0, then either A = 0 or B = 0. A Lie ring is a ring in which the multiplication is anticommutative, i.e., $x^2 = 0$ and the Jacobi identity (xy)z + (yz)x + (zx)y = 0, for all x,y,z in R is satisfied.

Main Results : Throughout this paper we consider a ring R with commutators and square of every element is in the nucleus.

i.e., $(R,R) \subset N$ ... (1)

and $x^2 \in N$, for all $x \in R$. ... (2)

If S(x,y,z) = (x,y,z) + (y,z,x) + (z,x,y), we have the identity

(xy,z) + (yz,x) + (zx,y) = S(x,y,z). ... (3)

Using (1) in the above identity, we have

$S(x,y,z) \subset N$.

i.e., $(x,y,z) + (y,z,x) + (z,x,y) \in N$, ... (4)

for all x, y, z in R.

Now we prove the following lemma.

Lemma 1 : Let R be a prime ring satisfying $x^2 \in N$ for every $x \in R$ and of char. ≠2. Then either R is associative or $N^2 = 0$.

Proof : For all r,s ∈ R,

$rs + sr = (r + s)^2 - r^2 - s^2$ must be in N.

Select n,n′ ∈ N, and x, y, z, ∈ R.

Then using the above equation

(n(n′x + xn′), y, z) = 0. So that

(nn′x,y,z) = - (n x n′,y,z).

Similarly (nxn′,y,z) = - (xnn′,y,z) and

(xnn′,y,z) = - (nn′x,y,z).

By combining the above three equalities it follows that 2 (nn′x,y,z) = 0. Since R is of char. ≠2, we get

(n n′x,y,z) = 0. ... (5)

Now (nx,y,z) = ((nx)y)z − (nx) (yz)
= (n(xy))z − n(x(yz))
= n((xy)z) − n(x(yz))
= n(x,y,z).

By replacing n by n n′ in the above equation, we get

(n n′x,y,z) = n n′(x,y,z). ... (6)

The combination of (5) and (6) yields n n′(x,y,z) = 0. So that

$N^2(R,R,R) = 0$. ... (7)

Let A be the ideal generated by N^2, and I the ideal generated by all associators (R,R,R). We have n n′x = n (n′x + x n′) - (nx + xn)n′ + xnn′. So that $N^2R \subset RN^2 + N^2$. Consequently

$A = R N^2 + N^2$.

In an arbitrary ring

I = (R,R,R) + (R,R,R)R. It follows from (7), that AI = 0. Since R is prime either

A = 0, or I = 0. If A = 0, then $N^2 = 0$. On the other hand if I = 0, then R is associative. This completes the proof of the lemma.

Theorem1. Let R be a prime ring of char. ≠2 satisfying $r^2 \in N$ for all $r \in R$. Then either R is associative or $r^2 = 0$, for all $r \in R$.

Proof : By considering the case N≠R, from the lemma 1 we have $N^2 = 0$.

Let K = N + NR.

Since rn = (rn + nr) − nr ∈ K and

snr = (sn + ns)r- nsr ∈ K, for all n ∈ N and r,s ∈R, K must be an ideal of R.

Moreover if

n′∈ N, nrn′ = n (rn′ + n′r) - nn′r = 0, since $N^2 = 0$. Therefore $K^2 = 0$. But R is prime and so K = 0. But then N = 0, whence $r^2=0$, for all r∈ R. This completes the proof of the theorem.

Theorem 2 : Let R be a prime ring of char. ≠2 satisfying (i) $x^2 \in N$ for all $x \in R$, and (ii) $(x,y,z) + (y,z,x) + (z,x,y) \in N$ for all $x,y,z \in R$. Then R is either associative or a Lie ring.

Proof : Assume that R satisfies (i) and (ii).

By considering the case $N \neq R$, from Theorem 1 we have $x^2 = 0$, for all $x \in R$. Consequently R is anti-commutative. i.e., $xy = -yx$.

For any $n \in N$, $n(xy) = (nx)y= (-xn)y= -x(ny)= x(yn)= (xy)n$,

And also $n(xy) = (-xy)n$.

Using the above two equations, we get $2n(xy) = 0$. Since R is of char. ≠2, we have $n(xy) = 0$. Thus $NR^2 = 0$. The set T of all $t \in R$ such that $tR = 0$, forms an ideal of R which must be zero since R is prime. Since $NR \subset T$, we obtain $NR = 0$ and subsequently $N = 0$.

In any anti-commutative ring $(x,y,z) + (y,z,x) + (z,x,,y) = (xy)z - x(yz) + (yz)x - y(zx)+ (zx)y - z(xy) = 2((xy)z + (yz)x + (zx)y)$ which equals twice the Jacobian of x, y, z. Then because of (ii), we have $2((xy)z + (yz)x+(zx)y) \in N$. Since R is of char. ≠2, we have $((xy)z + (yz)x + (zx)y) \in N$. Since $N = 0$, we conclde that the well known Jacobi identity holds and thus R is a Lie ring.

Now we give an example in which char. ≠2 is necessary in theorem 1.

Example 1 : Let 1, x, y be basis elements of the algebra R over an arbitrary field F, where $xy = 1$, $yx = x^2 = y^2 = 0$.

For any $\alpha, \beta, \gamma \in F$,

$$(\alpha + \beta x + \gamma y)^2 = \alpha^2 + 2\alpha\beta x + 2\alpha\gamma y + \beta\gamma$$
$$= 2\alpha (\alpha + \beta x + \gamma y) + \beta\gamma - \alpha^2.$$

Thus R is quadratic over F. Clearly R is simple, power associative, and that all commutators of R are contained in F. R is not associative since $(x,y,y) = y$. Also $(x+y)^2 = 1 \neq 0$. If F happens to be a field of char. = 2 then $r^2 \in F$ for every $r \in R$. Therefore Theorem 1 fails to hold for rings of char. = 2.

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