ASSOSYMMETRIC RINGS WITH $(X, (XY)^2) = 0$

DR. C. JAYA SUBBA REDDY, K. HEMAVATHI

Abstract : In this paper we prove that an assosymmetric ring satisfying $(x, (xy)^2) = 0$ or $(x, xy\text{-}y^2x^2) = 0$ is commutative.

Keywords : Assosymmetric, commutator,Associator,Nucles.

Introduction : M. Janjic [2] and Ashraf and Quadri [1] proved some commutativity theorems for associative rings with conditions like $(x, (xy)^n) = 0$ and $[x, xy - y^mx^n] = 0$. In this direction we prove the commutativity of non-associative assosymmetric ring with unity 1 satisfying any one of the following identities: (i) $(x, (xy)^2) = 0$ (ii) $(x, xy - y^2x^2) = 0$ (iii) $(x, xy^2 - y^4x) = 0$.

Preliminaries : The associator (x, y, z) is defined by $(x, y, z) = (xy) z - x(yz)$ for all x,y,z in a ring. The commutator (x, y) is defined by $(x, y) = xy - yx$ for all x, y in a ring. This can be considered to be a measure of non-commutativity of ring. By the nucleus N of a ring R, we mean the set of all elements n in R such that $(n, R, R) = (R, n, R) = (R, R, n) = 0$. A ring R is said to be of characteristic $\neq$ n if $n x = 0$ impels $x = 0$ for all $x \in R$ and n is a natural number. An assosymmetric ring R is one in which $(x, y, z) = (P(x), P(y), P(z))$, where P is any permutation of x, y, z in R.

Theorem 1: Let R be a 2-divisible assosymmetric ring with unity 1 such that
$[x, (xy)^2] = 0$ for all x, y in R. Then R is commutative and associative.

Proof: Let x, y be in R.
Then $[x, (xy)^2] = 0$.
That is
$x(xy)^2 - (xy)^2x = 0.$ 1
Replacing x by x + 1 in 1 and using 1 we get
$xy^2 + x ((xy) y) + x(y (xy)) - y^2x - ((xy) y) x - (y (xy)) x = 0.$ 2
Replacing x by x + 1 in 2 and using 2 we get
$2xy^2 - 2y^2x = 0.$
Thus $xy^2 - y^2x = 0.$ ...3
Replacing y by y + 1 in 3 and using 3 we get
$2xy - 2yx = 0.$
Thus $xy - yx = 0.$
That is, $xy = yx.$
Hence R is commutative.
In every assosymmetric ring we have the identity
$(xy, z) = x(y, z) + (x, z) y + (x, y, z).$
Since R is commutative, $(x, y, z) = 0.$
Therefore R is associative.
Similarly we can prove the following theorem

Theorem 2: Let R be a 2-divisible assosymmetric ring with unity 1 such that $[y, (xy)^2] = 0$ for all x, y in R. Then R is commutative and associative.

Theorem 3: Let R be a 2-divisible assosymmetric ring with unity 1 such that $[x, xy - y^2x^2] = 0$ for all x, y in R. Then R is commutative.

Proof: Let x, y be in R.
Then $[x, xy - y^2x^2] = 0.$
That is
$x (xy - y^2x^2) - (xy - y^2x^2)x = 0.$ ...4
Replacing x by x + 1 in 4 and using 4 we get
$xy - yx - xy^2 + y^2x - 2x(y^2x) + 2(y^2x)x = 0.$
....5
Replacing x by x + 1 in 5 and using 5 we get
$2y^2x - 2xy^2 = 0.$
Thus $y^2x - xy^2 = 0.$ 6
Replacing y by y + 1 in 6 and using 6 we get
$2yx - 2xy = 0.$
Thus, $yx - xy = 0.$
That is, $xy = yx.$
Hence R is commutative.
Similarly we can prove the following theorem.

Theorem 4: Let R be a 2-divisible assosymmetric ring with unity 1 such that $[y, xy - y^2x^2] = 0$ for all x, y in R. Then R is commutative.

Theorem 5: Let R be a 2- and 3-divisible assosymmetric ring with unity 1 such that $(x, xy^2 - y^4x) = 0$ for all x, y in R. Then R is commutative.

Proof: Let x, y be in R.
Then $(x, xy^2 - y^4x) = 0.$
That is
$x (xy^2 - y^4x) - (xy^2 - y^4x)x = 0.$ 7
Replacing x by x + 1 in 7 and using 7 we get
$xy^2 - xy^4 - y^2x + y^4x = 0.$ ...8
Replacing y by y + 1 in 8 and using 8 we get
$-2xy + 2yx - 6xy^2 + 6y^2x - 4xy^3 + 4y^3x = 0.$
Thus
$-xy + yx - 3xy^2 + 3y^2x - 2xy^3 + 2y^3x = 0.$ ...9
Replacing y by y + 1 in 9 and using 9 we get
$-12xy + 12yx - 9xy^2 + 9y^2x = 0.$
Thus
$-4xy + 4yx - 3xy^2 + 3y^2x = 0$ 10
Replacing y by y + 1 in 10 and using 10 we get
$-6xy + 6yx = 0.$
Thus, $-xy + yx = 0.$
That is, $xy = yx.$
Hence R is commutative.
Similarly we can prove the following theorem.

Theorem 6: Let R be a 2- and 3-divisible assosymmetric ring with unity 1 such that
$(x, xy^2 - y^4x^2) = 0$ for all x, y in R. Then R is commutative.

Example:Let $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} / a, b, c \text{ are integers} \right\}$
.
It can be easily verified that for any x, y in R, $[x, (xy)^2] = 0$ and $[x, xy - y^2x^2] = 0$.

However R is not commutative. Therefore unity 1 is essential in the hypothesis of the theorems.

References :

1. Ashraf, M and Quadri, M.A. " On commutativity of associative rings with constraints involving a subset", Rad, Mat.5 (1989), No. 1, 141-149.
2. Janjic, M "A note on the commutativity of rings", RADOVI MATHEMATICKI. Vol.2 (1987), 179-184.

* * *

Assistant Professor, e-mail:cjsreddysvu@gmail.com
Research Scholar, e-mail: hemavathiphd@gmail.com
Department of Mathematics, S.V.University, Tirupathi.