ASSOSYMMETRIC RINGS WITH WEAK NOVIKOV IDENTITY

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Abstract : In this paper the square of every element of an assosymmetric ring R is in the nucleus. Using this we prove that the nonzero idempotent e in a prime assosymmetric ring R is the identity element of R.
Keywords : Assosymmetric Ring, Nucleus, idempotent, prime assosymmetric ring.

Introduction : Right alternative rings satisfying the weak Novikov identity are studied in [2] and it is shown that the square of every element of the ring is in the nucleus. Paul [3] proved that if R is a prime non-associative ring satisfying (x, y, z) = (x, z, y) and with commutators in the left nucleus, then a non-zero idempotent e is the identity element of R if and only if e belongs to the nucleus. Now we prove that in this paper a non-associative 2- and 3-divisible prime assosymmetric ring R satisfying the weak Novikov identity, the square of every element of R is in the nucleus and the non-zero idempotent e in R is the identity element of R.

Preliminaries : The associator (x, y, z) is defined by (x, y, z) = (xy) z − x(yz) for all x,y,z in a ring.The commutator (x, y) is defined by (x, y) = xy − yx for all x, y in a ring. This can be considered to be a measure of non-commutativity of ring. By the nucleus N of a ring R, we mean the set of all elements n in R such that (n, R, R) = (R, n, R) = (R, R, n) = 0. A ring R is said to be of characteristic $\neq$ n if n x = 0 impels x = 0 for all $x \in R$ and n is a natural number. Rings satisfying the identity x(yz) = y(xz) are called Strongly Novikov and rings satisfying (w,x,yz) = y(w,x,z) are called Weakly Novikov. An assosymmetric ring R is one in which (x, y, z) = (P(x), P(y), P(z)), where P is any permutation of x, y, z in R.

In an arbitrary ring the following identities hold:

(wx, y, z) − (w, xy, z) + (w, x, yz) = w(x, y, z) + (w, x, y) z 1

f (w, x, y, z) = (wx, y, z) − x(w, y, z) − (x, y, z)w,

(x, y, z) + (y, z, x) + (z, x, y) = (xy, z) + (yz, x) + (zx, y). 2

(xy, z) − x(y, z) − (x, z)y = (x, y, z) − (x, z, y) + (z, x, y). 3

Putting z = x in 3 gives

(xy, x) + x(x, y) = (x, y, x). 4

In any assosymmetric ring 3 becomes

(xy, z) − x(y, z) − (x, z)y = (x, y, z) ...5

It is proved in [1] that in a 2- and 3-divisible assosymmetric ring R the following identities hold for all w, x, y, z, t in R:

f(w, x, y, z) = 0, that is, (wx, y, z) = x (w, y, z) + (x, y, z)w 6

((w, x), y, z) = 0 7

and ((w, x, y), z, t) = 0 8

That is, every commutator and associator is in the nucleus N. Suppose that $n \in N$. Then with w = n in 1 we get (nx, y, z) = n(x, y, z). Combining this with 7 yields

(nx, y, z) = n(x, y, z) = (xn, y, z). 9

From 7, we have $(R, N) \subseteq N$. 10

Lemma 1 : If R is a non-associative 2- and 3-divisible prime assosymmetric ring, then all commutators are in the center.

Proof: By forming associators on each side of 4 and u sing 7 gives

((x, y, x), r, s) = (x(x, y), r, s) = ((x, y) x, r, s).

Using 1 and 7, we have

((x, y) x, r, s) = (x, y) (x, r, s).

We conclude that

((x, y, x), r, s) = (x, y) (x, r, s).

By linearising this, we obtain ((x, y, t) + (t, y, x), r, s) = (x, y) (t, r, s) + (t, y) (x, r, s). If we substitute a commutator v for t, we see that (v, y) (x, r, s) = 0. This can be restated as

((R, R), R)(R, R, R) = 0. But now the ideal generated by double commutators ((R, R), R) annihilates the associator ideal. Since R is prime and not associative, we conclude that

((R, R), R) = 0 11

Thus the commutators are in the center.

By forming the commutators of each side of 2 with w and using 11 it follows that

3((x, y, z), w) = 0.

Thus ((x, y, z), w) = 0.

Theorem 1 : If R is a non-associative 2- and 3-divisible assosymmetric ring satisfying weak Novikov identity

(w, x, yz) = y(w, x, z), ...12

then x^2 is in the nucleus N.

Proof: By taking w = x in 6, we get

(x^2, y, z) = x(x, y, z) + (x, y, z) x. 13

In an assosymmetric ring we have

$(x^2, y, z) = (y, z, x^2)$.

On the other hand 12 implies that

(y, z, x^2) = x(y, z, x) = x(x, y, z).

Thus from 13 we must have (x, y, z) x = 0.

Since ((x, y, z), x) = 0 and (x, y, z) x = 0 we have x(x, y, z) = 0, so that using 13, we get $(x^2, y, z) = 0$.

Therefore x^2 is in the nucleus N.

Theorem 2: If R is a non-associative 2- and 3-divisible prime assosymmetric ring satisfying the weak Novikov identity, then the non-zero idempotent e in R is the identity element of R.

Proof: From theorem 1, $e \in N$. By lemma 1 in Rich [4], we have a decomposition $R = \oplus R_{ij}$, i, j = 0, 1, relative to e with $R_{ij}R_{kl} \subseteq \delta_{jk}R_{il}$ (δ denotes the Kronecker delta). Now R_{10} = (e, R_{10}) = - (R_{10}, e) and R_{01} = (R_{01}, e). Since $e \in N$ and $(R, N) \subseteq N$, R_{10} and R_{01} are contained in N. Now N is a subring or R. It follows that $R_{10}R_{01} + R_{01}R_{10} \subseteq N$. This, together with the property $R_{ij}R_{kj} \subseteq \delta_{jk}R_{il}$, allows us to conclude that

B = R_{10} R_{01} + R_{10} + R_{01} + R_{01} R_{10} is an ideal of R contained in N. Let I be the associator ideal of R. We shall show that BI = (0). Let $b \in B$.

Then using 1 we get (bx, y, z) – (b, xy, z) + (b, x, yz) = (b(x, y, z) + (b, x, y)z.

Since B is an ideal contained in N and $b \in B$ we have (bx, y, z) = (b, xy, z) = (b, x, yz) = (b, x, y)z = 0.

Thus, from the above equation, we get

b (x, y, z) = 0.

Also, since $b \in N$, b((x, y, z) w) = (b(x, y, z))w = 0.

Thus we have proved that bI = (0) for all b in B. Hence BI = (0). But R is prime and non-associative. This implies that B = (0). So we have $R = R_{11} \oplus R_{00}$. Thus, R_{11} and R_{00} are ideals of R such that $R_{11}R_{00}$ = (0). From the primeness of R again R_{11} = (0). R_{00} = (0). But $0 \neq e \in R_{11}$. So that $R_{11} \neq$ (0). We must have R_{00} = (0). This implies that e is the identity element of R.

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