NUMERICAL INVESTIGATION OF THE EFFECTS OF TWO-DIMENSIONAL CONVECTIVE FLOW PARAMETERS ON THE MARGINAL STABILITY

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Abstract : Here we present the effects on the marginal stability for a two dimensional convective flow in a horizontal mushy layer. We assume that this mushy layer has a permeable mush-liquid interface and variable permeability. The computational results presented here are based on the equations developed in the paper [12]. We compute the basic state solutions and the critical pair (Rayleigh number, wavenumber) using JMSL (Java Mathematical and Statistical Library) and fourth order Runge-Kutta method in combination of the shooting method.

Introduction : Analysis of stability of flows has been an important active research area in many applied fields including applied mathematics, various sciences and many branches of engineering. Stability related to hydrodynamics has been studied various authors [1,2]. There are several applications of porous medium convection [3]. Fowler [4] developed a mathematical model for the convective and freckling. During solidification of binary alloys, experimentalists observed a partially solidified layer consisting of dendrites. The mass and heat transfer within the mushy layer can cause impurity known as freckle. Many previous studies [5-12] have investigated in details about the mechanism of freckle formation. The objective of the present investigation is to analyze numerically the effects of various parameters in a reactive mushy layer.

2. Governing Equations for the System

We consider that the thickness of the mushy layer is d. Nondimensional governing equations representing the mushy layer system is given by [5, 7, 8, 9]

$$\mathrm{K}\,\vec{U} + \nabla P + R\,\Theta\,\hat{k} = 0,$$

(2.1)

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial z}\right)[(1-\Phi)\,\Theta + C\,\Phi] + \vec{U}\cdot\nabla\Theta = 0.$$

(2.2)

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial z}\right)[\Theta - S\,\Phi] + \vec{U}\cdot\nabla\,\Theta = \nabla^2\,\Theta\,,$$

(2.3)

$$\nabla\cdot\vec{U} = 0,$$

(2.4)

with boundary conditions become

Θ = −1, W = 0 at z = 0

Θ = Φ = ∂zW = 0 at z = δ

where $\vec{U}$, K, P, R, Θ, $\hat{k}$, t, z, Φ, C and S respectively represents the velocity, permeability of the medium, pressure, Rayleigh number, temperature, unit vector along vertical upward direction, time variable, space variable in vertical direction, solid fraction, concentration ratio and the Stefan number. Here W is the vertical component of $\vec{U}$ and δ denotes the thickness.

2.1 Solution Procedure

We write our system as following

$$\Theta = \theta_b(z) + \varepsilon\,\theta\,(x,y,z,t),$$
$$\Phi = \phi_b(z) + \varepsilon\,\phi\,(x,y,z,t),$$
$$\vec{U} = \vec{0} + \varepsilon\,\vec{u}\,(x,y,z,t),$$

(2.5)

$$P = p_b(z) + \varepsilon\,p\,(x,y,z,t),$$
$$\mathrm{K} = K_b(\phi_b) + \varepsilon\,K(\phi),$$

where $\theta_b, \phi_b, p_b, k_b$ are for the steady basic state motionless system, $\theta, \phi, \vec{u}, p$, K are corresponding perturbed variables. The perturbation parameter ε is given by $\varepsilon^2 = (R - R_c)/R_1$.

2.2 Steady State Solutions

Steady basic state solution for θ_b in implicit form is given by [10, 12]

$$z = \frac{\alpha_1 - C}{\alpha_1 - \beta_1}\ln\left[\frac{1+\alpha_1}{\alpha_1 - \theta_b}\right] + \frac{C - \beta_1}{\alpha_1 - \beta_1}\ln\left[\frac{1+\beta_1}{\beta_1 - \theta_b}\right]$$

(2.6)

Here α_1 and α_2 are, respectively, given by

$$\alpha_1 = \frac{C + S + \theta_\infty + \sqrt{(C+S+\theta_\infty)^2 - 4\,C\,\theta_\infty}}{2},$$

$$\alpha_2 = \frac{C + S + \theta_\infty - \sqrt{(C+S+\theta_\infty)^2 - 4\,C\,\theta_\infty}}{2},$$

where θ_∞ is the non-dimensional far field temperature. ϕ_b is obtained as

$$\phi_b = \frac{\theta_b}{\theta_b - C} \qquad (2.7)$$

Mushy layer thickness is obtained as

$$\delta = \frac{\alpha_1 - C}{\alpha_1 - \alpha_2} \ln\left[\frac{1+\alpha_1}{\alpha_1}\right] + \frac{C - \alpha_2}{\alpha_1 - \alpha_2} \ln\left[\frac{1+\alpha_2}{\alpha_2}\right]. \qquad (2.8)$$

2.3 Perturbed System

Here we follow a procedure mentioned by Chandrasekhar [1]. Introducing poloidal and toroidal components of $\vec{u}$ as u_P and u_T respectively and writing Δ_2 as the Laplacian in the xy- plane, for 2-dimensional case, perturbed system can be expressed as (see [9])

$$k_b \nabla^2 (\Delta_2 u_P) - R(\Delta_2 \theta) + \frac{\partial k_b}{\partial z}\frac{\partial}{\partial z}(\Delta_2 u_P) = -\left[\frac{\partial^2}{\partial x^2}\{K(\Delta_2 u_P)\} + \frac{\partial^2}{\partial x \partial z}\left\{K \frac{\partial^2 u_P}{\partial x \partial z}\right\}\right] \qquad (2.9)$$

$$\left(\nabla^2 + \frac{\partial}{\partial z} - \frac{\partial}{\partial t}\right)\theta - S\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial t}\right)\phi + (\Delta_2 u_P)\frac{\partial \theta_b}{\partial z} = \left[\frac{\partial^2 u_P}{\partial x \partial z}\frac{\partial \theta}{\partial x} - (\Delta_2 u_P)\frac{\partial \theta}{\partial z}\right] \qquad (2.10)$$

$$\left[\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial t}\right)\right][(\theta_b - C)\,\phi - (1-\phi_b)\theta] - (\Delta_2 u_P)\frac{\partial \theta_b}{\partial z} = -\left[\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial t}\right)(\theta\phi) + \frac{\partial^2 u_P}{\partial x \partial z}\frac{\partial \theta}{\partial x} - (\Delta_2 u_P)\frac{\partial \theta}{\partial z}\right] \qquad (2.11)$$

Linear System :

Now using the subscripts 0, 1 and 2 are used for the linear, first-order and second-order perturbed quantities respectively and expressing the dependent variables as

$$u_P(x,z,t) = \varepsilon^0 u_{P_0}(x,z,t) + \varepsilon^1 u_{P_1}(x,z,t) + \varepsilon^2 u_{P_2}(x,z,t) + \ldots\ldots$$
$$\theta(x,y,z,t) = \varepsilon^0 \theta_0(x,z,t) + \varepsilon^1 \theta_1(x,z,t) + \varepsilon^2 \theta_2(x,z,t) + \ldots\ldots \qquad (3.1)$$
$$\phi(x,y,z,t) = \varepsilon^0 \phi_0(x,z,t) + \varepsilon^1 \phi_1(x,z,t) + \varepsilon^2 \phi_2(x,z,t) + \ldots\ldots$$
$$K(x,y,z,t) = \varepsilon^0 K(x,z,t) + \varepsilon^1 K_1(x,z,t) + \varepsilon^2 K_2(x,z,t) + \ldots\ldots$$

the linear system can obtained as

$$k_b \nabla^2 (\Delta_2 u_{P_0}) - R_c(\Delta_2 \theta_0) + (\partial_z k_b)\partial_z(\Delta_2 u_{P_0}) = 0$$
$$(\nabla^2 + \partial_z - \partial_t)\theta_0 - S(\partial_z - \partial_t)\phi_0 + (\partial_z \theta_b)\partial_z(\Delta_2 u_{P_0}) = 0 \qquad (3.2)$$
$$(\partial_z - \partial_t)[(\theta_b - C)\phi_0 - (1-\phi_b)\theta_0] - (\partial_z \theta_b)\partial_z(\Delta_2 u_{P_0}) = 0$$

where u_{P_0}, θ_0, ϕ_0 are unknown linear dependent variables. The boundary conditions are

$u_{P_0} = \theta_0 = 0$ at z = 0 $\qquad$ $\partial_z u_{P_0} = \theta_0 = \phi_0 = 0$ at z = δ.

Using normal mode approach [1], and writing $f_0(x,z,t_s) = A(t_s)\tilde{f}(z)e^{i\alpha x} + C.C.$ where $C.C.$ stands for complex conjugate, the PDE system (3.2) becomes a linear ODE system

$$k_b\left[D^2 - \alpha^2\right]\tilde{u}_{P_0} - \mathcal{R}_c\tilde{\theta}_0 + (Dk_b)(D\tilde{u}_{P_0}) = 0 \qquad (3.3)$$

$$\left[D^2 + D - \alpha^2\right]\tilde{\theta}_0 - \mathcal{S}\left(D\tilde{\phi}_0\right) - (D\theta_b)\,\alpha^2\tilde{u}_{P_0} = 0 \qquad (3.4)$$

$$D\left[(\theta_b - \mathcal{C})\tilde{\phi}_0 - (1-\phi_b)\tilde{\theta}_0\right] + (D\theta_b)\,\alpha^2\tilde{u}_{P_0} = 0 \qquad (3.5)$$

where D = ∂_z . The adjoint system is given by (see [12])

$$\{k_b D^2 + (Dk_b)D - \alpha^2 k_b\}\,\hat{u}_P + \alpha^2\left(\hat{\phi} - \hat{\theta}\right)(D\theta_b) = 0$$
$$\{D^2 - D - \alpha^2\}\,\hat{\theta} - R_0\hat{u}_P + (1-\phi_b)\left(D\hat{\phi}\right) = 0 \qquad (3.6)$$
$$\mathcal{S}\left(D\hat{\theta}\right) + (\mathcal{C} - \theta_b)\left(D\hat{\phi}\right) = 0$$

Numerical Results : For variable permeability case, we choose n =3 and for constant permeability case, we choose n =0. We use C =5, 8 and S =1, 2. The thickness of the layer for various parameters is presented in table 1. Computed critical wavenumber, Rayleigh number are given in table 2. It is observed that critical Rayleigh number is higher for higher values of far field temperature.

Table 1: The thickness of the layer for different parameters

C	S	θ_∞	δ
5	1.0	0.1	2.15388
5	1.0	0.2	1.63401
5	1.0	0.3	1.35108
5	2.0	0.1	1.95989
5	2.0	0.2	1.50515
5	2.0	0.3	1.25510
8	1.0	0.1	2.23450
8	1.0	0.2	1.68627
8	2.0	0.1	2.09456
8	2.0	0.2	1.59427

Table 2: The critical pair (α_c, R_C) for various parameters

C	S	θ_∞	n	(α_c, R_C)
5	1.0	0.1	0	(1.21, 15.9038)
5	1.0	0.2	0	(1.53, 18.4395)
5	1.0	0.3	0	(1.81, 20.9515)
5	1.0	0.1	3	(1.25, 19.7393)
5	1.0	0.2	3	(1.58, 23.1963)
5	2.0	0.1	0	(1.35, 15.5320)
5	2.0	0.2	0	(1.67, 17.7472)
5	2.0	0.1	3	(1.38, 19.2350)
5	2.0	0.2	3	(1.72, 22.2658)
8	1.0	0.1	3	(1.19, 18.4975)

Figures 4.1 and 4.2 present results for marginal stability curves different θ_∞ and S = 1, C = 5 with n = 0 and n = 3 respectively. Comparison of the linear marginal stability curves with respect to the permeability is shown in figures 4.3 and 4.4. The critical wavenumber and critical Rayleigh number are higher for variable permeability case than the constant permeability case.

Figure 4.1: Marginal stability curves with n = 0 for different θ_∞ and S = 1.0, C = 5.0.

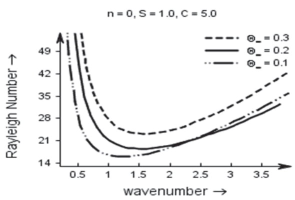


Figure 4.2: Marginal stability curves with n = 3 for different θ_∞ for S = 1, C = 5

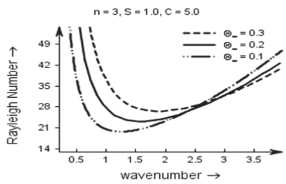


Figure 4.3: Marginal stability curves for S = 1.0, C =5.0.

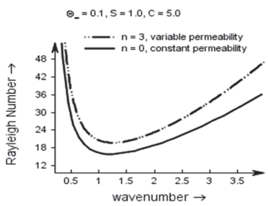
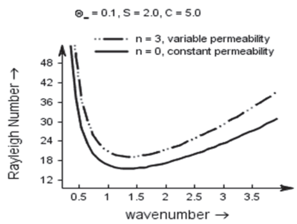


Figure 4.4: Marginal stability curves for S = 2.0, C =5.0.



Numerical results from figures 4.5 and 4.6 indicate that lower Stefan number improves the stability.

Figure 4.5: Marginal stability curves with different concentration ratios and θ_∞ = 0.1, S = 1.0.

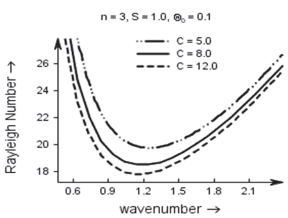
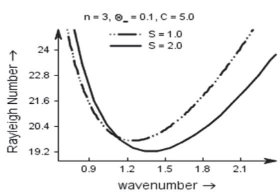


Figure 4.6: Marginal stability curves with different Stefan numbers with θ_∞ =0.1, C = 5.



Conclusion : The linear stability is enhanced by smaller values of C for fixed S and far field temperature. It is also observed that higher far field temperature enhances the linear stability.

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